

1 Q

question 13

Question 1.1 State the Alternating Series Test and prove that any infinite series satisfying its conditions is convergent.

The Alternating Series Test states that if $a_1 > a_2 > a_3 > \dots$ and $\lim_{n \rightarrow +\infty} a_n = 0$ then the infinite series $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n$ is convergent.

Proof: Let s_m be the partial sum $\sum_{n=1}^m (-1)^{n-1} a_n$.

$$\text{Now } s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}$$

$$\text{And } s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k} \quad \forall k \in \mathcal{N}$$

\implies the subsequence s_1, s_3, s_5, \dots is non-increasing.

\implies the subsequence s_2, s_4, s_6, \dots is non-decreasing.

$$\text{But } s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$$

\implies These subsequences are bounded and convergent. (Every non-increasing or decreasing bounded sequence is convergent), and

$$\lim_{n \rightarrow +\infty} s_{2k} = \lim_{n \rightarrow +\infty} s_{2k-1} = s$$

We claim that $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n = s$

Let $\epsilon > 0$ be given. Then $\exists K_1, K_2 \in \mathcal{N}$ s.t.

$$|s - s_{2k-1}| < \epsilon \text{ when } k \geq K_1$$

$$\text{and } |s - s_{2k}| < \epsilon \text{ when } k \geq K_2$$

Choose N s.t. $N \geq 2K_1 - 1$ and $N \geq 2K_2$

$$\implies |s - s_m| < \epsilon \text{ when } m \geq N$$

$$\implies \sum_{n=1}^{+\infty} (-1)^{n-1} a_n = \lim_{n \rightarrow +\infty} s_m = s$$

Therefore is convergent. ■

Question 1.2 Does the infinite series

$$\sum_{n=1}^{+\infty} \frac{\cos n\pi + \sin \frac{1}{2}n\pi}{n^2}$$

satisfy the conditions of the Alternating Series Test?

No it does not satisfy the conditions of the Alternating Series Test. But it does converge using the Direct Comparison Test.