Question 12

**Question 0.1** *Prove that the infinite series* $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$ *is convergent.*

The Alternating Series Test states that if $a_1 > a_2 > a_3 > \ldots$ and $lim_{n\to+\infty} a_n = 0$ then the infinite series $\sum_{n=1}^{+\infty}(-1)^{n-1}a_n$ is convergent.

Proof: Let $s_m$ be the partial sum $\sum_{n=1}^{m}(-1)^{n-1}a_n$.

$$Now\ s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}$$

$$And\ s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k}\ \forall\ k \in \mathcal{N}$$

$\Longrightarrow$ the subsequence $s_1, s_3, s_5, \ldots$ is non-increasing.
$\Longrightarrow$ the subsequence $s_2, s_4, s_6, \ldots$ is non-decreasing.

$$But\ s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$$

$\Longrightarrow$ These subsequences are bounded and convergent. (Every non-increasing or decreasing bounded sequence is convergent), and

$$\lim_{n\to+\infty} s_{2k} = \lim_{n\to+\infty} s_{2k-1} = s$$

We claim that $\sum_{n=1}^{+\infty}(-1)^{n-1}a_n = s$

Let $\epsilon > 0$ be given. Then $\exists\ K_1, K_2 \in\ \mathcal{N}$ s.t.

$$|s - s_{2k-1}| < \epsilon\ when\ k\ \geq K_1$$

$$and\ |s - s_{2k}| < \epsilon\ when\ k\ \geq K_2$$

Choose $N$ s.t. $N \geq$ "$K_1 - 1$ and $N \geq 2K_2$

$$\Longrightarrow |s - s_m| < \epsilon\ when\ m\ \geq N$$

$$\Longrightarrow \sum_{n=1}^{+\infty}(-1)^{n-1}a_n = \lim_{n\to+\infty} s_m = s$$

Therefore is convergent. ∎

Using this result and since in the case of $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$ where $a_n > a_{n-1}\ \forall\ n$ and $\lim_{n\to+\infty} a_n = 0$ this series converge.