Question 11

Question 0.1 *Prove that the infinite series* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is divergent.*

Let s_m be the partial sum $\sum_{n=1}^{m} \frac{1}{n}$
We claim that $s_{2k} \geq \frac{k+1}{2}\ \forall\ k \in \mathcal{N}$.
When $k = 1$ the result is clear.
Now suppose that $s_{2k-1} \geq \frac{k}{2}$ then

$$s_{2k} = s_{2k-1} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \ldots + \ldots + \frac{1}{2^k} \geq s_{2k-1} + 2^{k-1}\left(\frac{1}{2^k}\right) = s_{k-1} + \frac{1}{2}$$

Therefore by induction on k it follows that $s_{2k} \geq \frac{k+1}{2}$
$\Longrightarrow$ the series $s_1, s_2, s_3, \ldots$ is not bounded and therefore cannot converge.
$\Longrightarrow$ the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent. ∎

Question 0.2 *Prove that the infinite series* $\sum_{n=2}^{+\infty} \frac{1}{nlogn}$

Let $a_n = \frac{1}{nlogn}$ and let $s_n = \sum_{m=2}^{n} \frac{1}{nlogn}$

$$\Longrightarrow s_{2^k} = \frac{1}{2log2} + \left(\frac{1}{3log3} + \frac{1}{4log4}\right) + \left(\frac{1}{5log5} + \frac{1}{6log6} + \frac{1}{7log7} + \frac{1}{8log8}\right)$$

$$+ \ldots + \left(\frac{1}{(2^{k-1}+1)log2^{k-1}} + \ldots + \frac{1}{2^k log2^k}\right) \geq \frac{1}{2log2} + \frac{2}{2log4} + \frac{4}{8log8}$$

$$+ \ldots + \frac{2^{k-1}}{2^k log2^k} = \frac{1}{2log2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{k}\right)$$

$\Longrightarrow\ s_{2^k} \geq \frac{1}{2log2} \sum_{n=2}^{+\infty} \frac{1}{n}$
Therefore $s_{2^k} \to +\infty$ as $n \to +\infty$ so the series is divergent. ∎.