Question 1

Question 0.1 *State and prove Rolles Theorem*

(Rolles Theorem) Let f be a continuous real valued function defined on some interval $[a, b]$ & differentiable on all (a, b). If $f(a) = f(b) = 0$ then $\exists$ some $s \in [a, b]$ s.t. $f'(s) = 0$.

f is continuous on $[a, b]$ therefore assumes absolute max and min values on $[a, b]$. These can only occur at :

1. Points on $[a, b]$ where $f'(x)$ doesn't exist.
2. The end points $a \& b$.
3. Some internal point s where $f'(s) = 0$

1. Void by hypothesis (f is continuous).
2. If either the end points $a \& b$ are a max or min then f is a constant function and s can be taken anywhere in $[a, b]$.
3. If a max or min occurs at some internal point s in $[a, b]$ then $f'(s) = 0$ and we have a point for the theorem. $\blacksquare$

Question 0.2 *Let* $f : \mathcal{R} \Longrightarrow \mathcal{R}$ *be a function which is* $2k = 1$ *times differentiable, for some non-negative integer* k. *Let* $a \& b$ $(a < b)$ & $(a, b \in \mathcal{R}$. *Suppose that* $f^{(j)}(A) = 0$ *and* $f^{(j)}(b) = 0$ *for* $j = 0, 1, \ldots, k$.*Prove that* $\exists$ *some* $\xi \in \mathcal{R}$ *s.t.* $a < \xi < b$ *for which* $f^{(2k+1)}(\xi) = 0$.

Suppose $0 \leq \ j \leq \ k$ & $f^{(j)}(x) = 0$ for $x = a_0, a_1, \ldots, a_{j+1}$ & $a = a_0 < a_1 < \ldots < a_{j+1} = b$

Applying Rolles Theorem to $f^{(j)}$ *on* $[a_{i-1}, a_i]$ for $i = 1, 2, \ldots, j+1$ $\exists$ b_i *s.t.* $a_{i-1} < b_i < a_i$ and s.t. $f^{j=1}(b_i) = 0$

Also $\exists$ $f^{j+1}(b_{j+2})$ *where* $b_{j+2} = b$

Repeatedly applying this result for $j = 0, 1, \ldots, k-1$ we see that $\exists$ $c_0, c_1, \ldots, c_{k+1}$ where $a = c_0 < c_1, < \ldots < c_{k+1} = b$ s.t. $f_{(k)}(x) = 0$ when $x = c_i$ $(i = 0, 1, 2, \ldots, k+1)$

$$So\ f^{(k)} = 0\ for\ k+2\ values\ in\ [a, b].$$

Applying Rolles Theorem to $f^{(k)}$ on the relevant subintervals in $[a, b]$

$$f^{(k+1)} = 0\ for\ k+1\ values\ in\ [a, b]$$

$$f^{(k+j)} = 0\ for\ k+2\ values\ in\ [a, b]\ for\ j = 0, 1, \ldots, k+1$$

In particular $f^{(2k+1)}(\xi) = 0\ for\ some\ \xi \in\ [a, b]$ $\blacksquare$

Question 0.3 *State the Mean Value Theorem, and show how it mad be deduced as a corollary of Rolles Theorem..*

(Mean Value Theorem) Let f be a continuous real valued function defined on some interval $[a, b]$ & differentiable on all (a, b). Then $\exists$ some $s \in [a, b]$ s.t. $a < s < b$ & $\frac{f(b)-f(a)}{b-a} = f'(s)$
Let $g(x) : [a, b] \Longrightarrow R$ be defined by

$$g(x) = f(x) - \frac{b-x}{b-a} f'(b) - \frac{x-a}{b-a} f'(a)$$

Applying Rolles Theorem to g on $[a, b]$ $\exists$ some $s \in [a, b]$ s.t. $g'(s) = 0$

$$\Rightarrow g'(s) = f'(s) - \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} = f'(s)$$

as required ∎