Question 1

Question 0.1 *State and prove Rolles Theorem*

(Rolles Theorem) Let f be a continuous real valued function defined on some interval $[a, b]$ & differentiable on all (a, b). If $f(a) = f(b) = 0$ then $\exists$ some $s \in [a, b]$ s.t. $f'(s) = 0$.

f is continuous on $[a, b]$ therefore assumes absolute max and min values on $[a, b]$. These can only occur at :

1. Points on $[a, b]$ where $f'(x)$ doesn't exist.
2. The end points $a \& b$.
3. Some internal point s where $f'(s) = 0$

1. Void by hypothesis (f is continuous).
2. If either the end points $a \& b$ are a max or min then f is a constant function and s can be taken anywhere in $[a, b]$.
3. If a max or min occurs at some internal point s in $[a, b]$ then $f'(s) = 0$ and we have a point for the theorem. $\blacksquare$

Question 0.2 *Let $f : \mathcal{R} \Longrightarrow \mathcal{R}$ be a function which is $2k = 1$ times differentiable, for some non-negative integer k. Let $a\&b$ $(a < b)$ & $(a, b \in \mathcal{R}$. Suppose that $f^{(j)}(A) = 0$ and $f^{(j)}(b) = 0$ for $j = 0, 1, \ldots, k$.Prove that $\exists$ some $\xi \in \mathcal{R}$ s.t. $a < \xi < b$ for which $f^{(2k+1)}(\xi) = 0$.*

Suppose $0 \leq\ j \leq\ k$ & $f^{(j)}(x) = 0$ for $x = a_0, a_1, \ldots, a_{j+1}$ & $a = a_0 < a_1 < \ldots < a_{j+1} = b$

Applying Rolles Theorem to $f^{(j)}$ *on* $[a_{i-1}, a_i]$ for $i = 1, 2, \ldots, j{+}1$ $\exists$ b_i *s.t.* $a_{i-1} < b_i < a_i$ and s.t. $f^{j=1}(b_i) = 0$

Also $\exists$ $f^{j+1}(b_{j+2})$ *where* $b_{j+2} = b$

Repeatedly applying this result for $j = 0, 1, \ldots, k - 1$ we see that $\exists$ $c_0, c_1, \ldots, c_{k+1}$ where $a = c_0 < c_1, < \ldots < c_{k+1} = b$ s.t. $f_{(k)}(x) = 0$ when $x = c_i$ $(i = 0, 1, 2, \ldots, k + 1)$

$$So\ f^{(k)} = 0\ for\ k + 2\ values\ in\ [a, b].$$

Applying Rolles Theorem to $f^{(k)}$ on the relevant subintervals in $[a, b]$

$$f^{(k+1)} = 0\ for\ k + 1\ values\ in\ [a, b]$$

$$f^{(k+j)} = 0\ for\ k + 2\ values\ in\ [a, b]\ for\ j = 0, 1, \ldots, k + 1$$

In particular $f^{(2k+1)}(\xi) = 0$ *for some* $\xi \in\ [a, b]$ $\blacksquare$

Question 0.3 *State the Mean Value Theorem, and show how it mad be deduced as a corollary of Rolles Theorem..*

(Mean Value Theorem) Let f be a continuous real valued function defined on some interval $[a, b]$ & differentiable on all (a, b). Then $\exists$ some $s \in [a, b]$ s.t. $a < s < b$ & $\frac{f(b)-f(a)}{b-a} = f'(s)$
Let $g(x) : [a, b] \Longrightarrow R$ be defined by

$$g(x) = f(x) - \frac{b-x}{b-a} f'(b) - \frac{x-a}{b-a} f'(a)$$

Applying Rolles Theorem to g on $[a, b]$ $\exists$ some $s \in [a, b]$ s.t. $g'(s) = 0$

$$\Rightarrow g'(s) = f'(s) - \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{f(b) - f(a)}{b-a} = f'(s)$$

as required ∎

Question 2

Let $f : [a,b] \Longrightarrow \mathcal{R}$ be a bounded function defined on the interval $[a,b]$

Question 0.1 *Define the concept of a partition of the interval* $[a,b]$. *Give the definition of lower sum* $L(P,f)$ & *upper sum* $U(P,f)$ *of* f *for partition* P

A Partition P is a set $\{x_1, x_2, x_3, \dots x_n\}$ $all \in \mathcal{R}$ s.t. $a = x_0 < x_1 < x_2 < \dots < x_n = b$.
The *lower sum* $L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$
The *upper sum* $U(P,f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$

$Where\ m_i = inf\{f(x) : x_{i-1} < x < x_i\}$ & $M_i = sup\{f(x) : x_{i-1} < x < x_i\}$

Question 0.2 *Define the lower Riemann Integral* $\mathcal{L}\int_a^b f(t)dt$ *and the upper Riemann Integral* $\mathcal{U}\int$ *of* f *on interval* $[a,b]$. *Define precisely what is meant by saying that the function* f *is Riemann Integrable on* $[a,b]$, *and define the Riemann Integral of a Riemann Integrable function on* $[a,b]$.

$$\mathcal{L}\int_a^b f(t)dt = sup(L(P,f))$$

$$\mathcal{U}\int_a^b f(t)dt = inf(U(P,f))$$

A function f on $[a,b]$ is Riemann integrable if $\mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt$.
The Riemann Integral of f on $[a,b]$ is the common value of $\mathcal{L}\int_a^b f(t)dt$ and $\mathcal{U}\int_a^b f(t)dt$.

Question 0.3 *Explain why* $L(P,f) \leq U(Q,f)$ *for all partitions* P *and* Q *on* $[a,b]$. *and show that*

$$\mathcal{L}\int_a^b f(t)dt \leq \mathcal{U}\int_a^b f(t)dt$$

Let R be a common refinement of P and Q. Using the fact that

$$L(P,f) \leq\ L(R,f) \leq\ U(R,f) \leq\ U(P,f)$$

$$And\ L(Q,f) \leq\ L(R,f) \leq\ U(R,f) \leq\ U(Q,f)$$

It is clear from the above inequalities that $L(P,f) \leq\ U(Q,f)$
Taking the *sup* of the L.H.S. of this inequality we get

$$\mathcal{L}\int_a^b f(t)dt \leq\ U(Q,f)$$

and taking the inf of the R.H.S. of this inequality we see that

$$\mathcal{L}\int_a^b f(t)dt \leq\ \mathcal{U}\int_a^b f(t)dt \qquad \blacksquare$$

Question 0.4 *Let* $f : [a, b] \Longrightarrow \mathcal{R}$ *be a constant function with value* c*. What are the values of* $L(P, f)$ *&* $U(P, f)$ *for any partition* P *on* $[a, b]$*.*

Now f is a constant function. Therefore $L(P, f) = U(P, f)$. Using the definitions from above it is clear that $m_i = M_i$ and the value is c.

$$\Longrightarrow L(P, f) = U(P, f) = \sum_{i=1}^{n}(x_i - x_{i-1}).c \; = \; (b - a).c$$

Question 0.5 *Let* $f : [0, 1] \Longrightarrow \mathcal{R}$ *be defined by*

$$f(t) = \{t \; if \; t \; is \; rational, 0 \; if \; t \; is \; irrational\}$$

Is f *Riemann Integrable on* $[a, b]$*? Justify your answer.*

f is not Riemann Integrable on $[a, b]$ because
Each interval $[x_i, x_{i-1}]$ contains both rational and irrational numbers

$$\Longrightarrow U(P, f) = t \; and \; L(P, f) = 0$$

$$\Longrightarrow \mathcal{L}\int_a^b f(t)dt \neq \mathcal{U}\int_a^b f(t)dt$$

Question 3
Let $f : [a, b] \Longrightarrow \mathcal{R}$ be a bounded function defined on the interval $[a, b]$

Question 0.1 *Define the concept of a partition of the interval* $[a, b]$. *Give the definition of lower sum* $L(P, f)$ *& upper sum* $U(P, f)$ *of* f *for partition* P

A Partition P is a set $\{x_1, x_2, x_3, \ldots x_n\}$ *all* $\in \mathcal{R}$ s.t. $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.
The *lower sum* $L(P, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$
The *upper sum* $U(P, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$

Where $m_i = inf\{f(x) : x_{i-1} < x < x_i\}$ & $M_i = sup\{f(x) : x_{i-1} < x < x_i\}$

Question 0.2 *Define the lower Riemann Integral* $\mathcal{L}\int_a^b f(t)dt$ *and the upper Riemann Integral* $\mathcal{U}\int$ *of* f *on interval* $[a, b]$. *Define precisely what is meant by saying that the function* f *is Riemann Integrable on* $[a, b]$, *and define the Riemann Integral of a Riemann Integrable function on* $[a, b]$.

$$\mathcal{L}\int_a^b f(t)dt = sup(L(P, f))$$

$$\mathcal{U}\int_a^b f(t)dt = inf(U(P, f))$$

A function f on $[a, b]$ is Riemann integrable if $\mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt$.
The Riemann Integral of f on $[a, b]$ is the common value of $\mathcal{L}\int_a^b f(t)dt$ and $\mathcal{U}\int_a^b f(t)dt$.

Question 0.3 *Let* $f : [0, 1] \Longrightarrow \mathcal{R}$ *be defined by* $f(t) = 1 - t^2$. *Calculate* $L(P_n, f)$ *and* $U(P_n, f)$, *where* P_n *denotes the partition of* $[0, 1]$ *into* n *subintervals of length* $\frac{1}{n}$. *ie.* $P = \{t_0, t_1, \ldots, t_n\}$ *where* $t_i = \frac{1}{n}$ *for* $i = 0, 1, \ldots, n$. *Hence show that*

$$\mathcal{L}\int_a^b f(t)dt = (U)\int_a^b f(t)dt = \frac{2}{3}$$

f takes values between $1 - \frac{(i-1)^2}{n}$ and $1 - \frac{i^2}{n}$ and $1 - \frac{i^2}{n} \leq 1 - \frac{(i-1)^2}{n}$
Using the definitions from above.

$$m_i = 1 - \frac{i^2}{n} \; and \; M_i = 1 - \frac{(i-1)^2}{n}$$

$$Now \; L(P_n, f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{i^2}{n})$$

$$\textit{And } U(P_n, f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \frac{1}{n}\sum_{i=1}^{n}(1 - \frac{(i-1)^2}{n})$$

$$\Longrightarrow L(P_n, f) = (1 - \frac{1}{n^3})\sum_{i=1}^{n} i^2 = \frac{1}{6n^2}(n+1)(2n+1) = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2}$$

$$\Longrightarrow U(P_n, f) = (1 - \frac{1}{n^3})\sum_{i=1}^{n-1} i^2 = \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$$

Then $\frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2} \leq \mathcal{L}\int_a^b f(t)dt \leq \mathcal{U}\int_a^b f(t)dt \leq \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2}$
But The upper and lower Riemann Integrals do not depend on n because it is constant.

$$\Longrightarrow \mathcal{L}\int_a^b f(t)dt = \mathcal{U}\int_a^b f(t)dt = \frac{2}{3} \qquad \blacksquare$$

Question 4.

Question 0.1 *Let* $f : [a,b] \Longrightarrow \mathcal{R}$ *be a continuous function defined on the interval* $[a,b]$ *and let*

$$F(x) = \int_a^x f(t)dt, (a \leq x \leq b)$$

Prove that $F('x) = f(x)$ *for all* x *s.t.*$(a < x < b)$

f is continuous $\Longrightarrow$ given any $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t. $|f(t) - f(x)| < \frac{1}{2}\epsilon$ $\forall\ t, x\ \in\ [a,b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) = \frac{1}{h}\int_x^{x+h}(f(t) - f(x))dt$$

If $0 < |h| < 1$ and $x + h \in\ [a,b]$. Then

$$\left|\int_x^{x+h}(f(t) - f(x))dt\right| < \frac{1}{2}\epsilon|h|$$

$$\Longrightarrow \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{1}{2}\epsilon < \epsilon$$

$$\Longrightarrow F'(x) = \lim_{h \to 0}\frac{F(x+h) - F(x)}{h} = f(x) \quad \blacksquare$$

Question 0.2 *Use the method for integration by parts to evaluate*

$$\int_0^s t^3 e^{-t} dt \ for\ any\ s > 0$$

note $\int udv = uv - \int vdu$

Let $u_1 = t^3 \implies du_1 = 3t^2dt$ and let $v_1 = -e^{-t} \implies dv_1 = e^{-t}dt$

$$\int_0^s t^3e^{-t}dt = -e^{-t}t^3 - \int_0^s(-e^{-t})3t^2dt$$

Let $u_2 = 3t^2 \implies du_2 = 6tdt$ and let $v_2 = -e^{-t} \Longrightarrow dv_2 = e^{-t}dt$

$$\int_0^s(-e^{-t})3t^2dt = -3t^2e^{-t} - \int_0^s(-e^{-t})6tdt$$

Let $u_3 = 6t \implies du_3 = 6dt$ and let $v_3 = -e^{-t} \implies dv_3 = e^{-t}dt$

$$\int_0^s -e^{-t})6tdt = -6te^{-t} + 6\int_0^s(-e^{-t})dt = -6te^{-t} - 6e^{-t}$$

$$\Longrightarrow \int_0^s(-e^{-t})3t^2dt = -3t^2e^{-t} - 6te^{-t} - 6e^{-t}$$

$$\Longrightarrow \int_0^s t^3e^{-t}dt = [-e^{-t}t^3 - 3t^2e^{-t} - 6te^{-t} - 6e^{-t}]_0^s$$

$$\Longrightarrow \int_0^s t^3e^{-t}dt = -e^{-t}(s^3 + 3s^2 + 6s + 6) + 6$$

Question 5.

Question 0.1 *Let* $f : [a,b] \Longrightarrow \mathcal{R}$ *be a continuous function defined on the interval* $[a,b]$ *and let*

$$F(x) = \int_a^x f(t)dt, (a \leq x \leq b)$$

Prove that $F('x) = f(x)$ *for all* x *s.t.*$(a < x < b)$

f is continuous $\Longrightarrow$ given any $\epsilon > 0$ $\exists$ some $\delta > 0$ s.t. $|f(t) - f(x)| < \frac{1}{2}\epsilon$ $\forall$ $t, x \in [a,b]$ satisfying $|t - x| < \delta$. Now

$$\frac{F(x+h) - F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x) = \frac{1}{h}\int_x^{x+h}(f(t) - f(x))dt$$

If $0 < |h| < 1$ and $x + h \in [a,b]$. Then

$$\left|\int_x^{x+h}(f(t) - f(x))dt\right| < \frac{1}{2}\epsilon|h|$$

$$\Longrightarrow \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| \leq \frac{1}{2}\epsilon < \epsilon$$

$$\Longrightarrow F'(x) = \lim_{h \to 0}\frac{F(x+h) - F(x)}{h} = f(x) \qquad \blacksquare$$

Question 0.2 *Find the Derrivative of the function* $g : \mathcal{R} \Longrightarrow \mathcal{R}$ *defined by*

$$g(x) = \int_0^{x^4} t^2 e^{t^2}$$

Using the fundemental theorem of calculus from above we know that if $F(x) = \int_a^x f(t)dt$ then $F'(x) = f(x)$
Now in this case

$$F(x) = \int_0^{x^4} t^2 e^{t^2} = F(t^2 e^{t^2})$$

$$F'(x) = 4x^{11}e^{x^8}$$

Question 6

Question 0.1 *Let f be a function that is $k-$times differentiable and whose k^{th} derrivative is continuous on some open interval containing $a, a+h$ where $a, h \in \mathcal{R}$ Using the Rule for integration by parts show that*

$$f(a+h) = f(s) + \sum_{n=1}^{n} \frac{t^n}{n!} f^{(n)}(s) + r_k(a,h)$$

$$Where\ r_k(a,h) = \frac{h^k}{(k-1)!} \int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt$$

Take the expression for $r_k(a,h)$ and work on it integrating by parts.

$$\int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt = \int_0^1 udv$$

Where $u = f^{(k)}(a+th)\ and\ du = hf^{(k+1)}(a+th)dt$
And $v = \frac{-1}{k}(1-t)^k\ and\ dv = (1-t)^{k-1}dt$
note that $\int udv = uv - \int vdu$ Then

$$\int_0^1 (1-t)^{k-1} f^{(k)}(a+th)dt = [f^{(k)}(a+th)\frac{-1}{k}(1-t)^k]_0^1 - \int_0^1 \frac{-1}{k}(1-t)^k h f^{(k+1)}(a+th)dt$$

$$= \frac{1}{k} f^{(k)}(a) + \frac{h}{k}\int_0^1 (1-t)^k f^{(k+1)}(a+th)dt\ (k>0)$$

$$\implies r_k(a,h) = \frac{h^k}{(k-1)!}\left(\frac{1}{k} f^{(k)}(a) + \frac{h}{k}\int_0^1 (1-t)^k f^{(k+1)}(a+th)dt\right)$$

$$= \frac{h^k}{k!} f^{(k)}(a) + r_{k+1}(a,h)$$

$$\implies\ r_1(a,h) = h\int_0^1 f'(a+th)dt = \int_0^1 \frac{d}{dt}(f(a+th))dt = [f(a+th)]_0^1$$

$$\implies f(a+h) = f(a) + r_1(a,h)$$

The required formula follows exactly by induction on k ▮

Question 7

Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous real-valued functions on an interval $[a, b]$ and let f be a real-valued function on $[a, b]$.

Question 0.1 *Define what is meant by saying that the functions f_n converge uniformly to f on the interval $[a, b]$ as $n \to \infty$*

The functions f_n are said to converge uniformly to some real-valued function f if given any $\epsilon > 0$ $\exists$ some $N \in \mathcal{N}$ s.t. $|f_n(x) - f(x)| < \epsilon$ $\forall$ $n \geq N$ and $\forall$ $x \in [a, b]$.

Question 0.2 *Suppose that the function f_n converges uniformly to f on $[a, b]$ as $n \to +\infty$. Prove that*

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$$

Let $\epsilon > 0$ be given. Choose some ϵ_0 s.t. $0 < \epsilon_0(b - a) < \epsilon$. Now

$$-\int_a^b |f_n(t) - f(t)|dt \leq \int_a^b f_n(t)dt - \int_a^b f(t)dt \leq \int_a^b |f_n(t) - f(t)|dt$$

now because f_n converges uniformly to f on $[a, b]$ then

$$\left|\int_a^b f_n(t)dt - \int_a^b f(t)dt\right| \leq \int_a^b |f_n(t) - f(t)|dt \leq \epsilon_0(b-a) < \epsilon \textit{ whenever } n \geq N$$

The Result follows (because ϵ can be chosen arbitrarily small) ∎

Question 0.3 *Give an example of a sequence $f_1, f_2, f_3 \ldots$ of real valued functions on an interval $[a, b]$ and a continuous real valued function f on $[a, b]$ s.t.*

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt \neq \int_a^b f(t)dt$$

even though $\lim_{n\to+\infty} f_n(t) = f(t)$ *for all* $t \in [a, b]$

Question 8

Question 0.1 *What is meant by saying that an infinite sequence $z_1, z_2, z_3 \ldots$ of complex numbers is convergent.*

An infinite sequence of complex numbers $z_1, z_2, z_3 \ldots$ is said to converge to some $l \in \mathcal{C}$ if given any $\epsilon > 0 \ \exists$ some $N \in \mathcal{N}$ s.t. $|z_n - l| < \epsilon \ \forall \ n \ \geq \ N$.

Question 0.2 *Suppose that $z_n = x_n + iy_n$ for all $n \in \mathcal{N}$ and $i = \sqrt{-1}$. Suppose the sequence $x_1, x_2, x_3, \ldots$ converges to λ and that the sequence $y_1, y_2, y_3, \ldots$ converges to μ where $\lambda, \mu \in \mathcal{R}$. Prove that the sequence $z_1, z_2, z_3, \ldots$ converges to the complex number $\lambda + i\mu$*

Suppose that $z_n \rightarrow l$ as $n \rightarrow +\infty$ where $l = \lambda + i\mu$. Then given any $\epsilon > 0 \ \exists \ some \ N \in \mathcal{N}$ s.t. $|z_n - l| < \epsilon \ \forall \ n \ \geq \ N$.
$\Longrightarrow |x_n - \lambda| < \epsilon \ \& \ |y_n - \mu| < epsilon \ \forall \ n \ \geq \ N$
Therefore $x_n \rightarrow \lambda$ and $y_n \rightarrow \mu$ as $n \rightarrow +\infty$.

Conversely suppose that $x_n \rightarrow \lambda$ and $y_n \rightarrow \mu$ as $n \rightarrow +\infty$ then given any $\epsilon > 0 \ \exists$ some $N_1, N_2 \in \mathcal{N}$ s.t. $|x_n - \lambda| < \frac{1}{\sqrt{2}}\epsilon$ when $n \geq N_1$ and $|y_n - \mu| < \frac{1}{\sqrt{2}}\epsilon$ when $n \geq N_2$.
Let $N = max(N_1, N_2)$ therefore when $n \geq N$ then

$$|z_n - l|^2 = |x_n - \lambda|^2 + |y_n - \mu|^2 \leq \frac{1}{2}\epsilon^2 + \frac{1}{2}\epsilon^2 + \epsilon^2$$

$$\Longrightarrow |z_n - l| < \epsilon \ as \ required \qquad \blacksquare$$

Question 0.3 *Let (z_n) and (w_n) be convergent infinite sequences of complex numbers. Prove that the infinite sequence $(z_n + w_n)$ is convergent.*

Suppose $z_n \rightarrow l_1$ as $n \rightarrow +\infty$ and $w_n \rightarrow l_2$ as $n \rightarrow +\infty$. Where $z_n = x_{n_1} + iy_{n_1}$ and $w_n = x_{n_2} + iy_{n_2}$. And $l_1 = \lambda_1 + i\mu_1$ and $l_2 = \lambda + i\mu_2$.
Then given any $\epsilon > 0 \ \exists$ some $N_1, N_2 \in \mathcal{N}$ s.t. $|z_n - l_1| < \epsilon$ when $n \geq N_1$ and $|w_n - l_2| < \epsilon$ when $n \geq N_2$

$$\Longrightarrow |x_{n_1} - \lambda_1| < \epsilon \ \& \ |y_{n_1} - \mu_1| < \epsilon \ when \ n \geq \ N_1$$

$$\Longrightarrow |x_{n_2} - \lambda_2| < \epsilon \ \& \ |y_{n_2} - \mu_2| < \epsilon \ when \ n \geq \ N_2$$

$$\Longrightarrow \ x_{n_1} \rightarrow \lambda_1 \ , \ y_{n_1} \rightarrow \mu_1 \ , \ x_{n_2} \rightarrow \lambda_2 \ , \ y_{n_2} \rightarrow \mu_2 \ all \ as \ n \rightarrow +\infty$$

Conversely suppose that $x_{n_1} \rightarrow \lambda_1$, $y_{n_1} \rightarrow \mu_1$, $x_{n_2} \rightarrow \lambda_2$, $y_{n_2} \rightarrow \mu_2$ *all as* $n \rightarrow +\infty$.

Let $\epsilon > 0$ be given then $\exists\ N_1, N_2, N_3, N_4 \in \mathcal{N}$ s.t.
$|x_{n_1} - \lambda_1| < \frac{1}{4}\epsilon$ when $n \geq N_1$, $|x_{n_2} - \lambda_2| < \frac{1}{4}\epsilon$ when $n \geq N_2$,
$|y_{n_1} - \mu_1| < \frac{1}{4}\epsilon$ when $n \geq N_3$ and $|y_{n_2} - \mu_2| < \frac{1}{4}\epsilon$ when $n \geq N_4$. Let $N = max(N_1, N_2, N_3, N_4)$ then

$$|z_n - l_1| + |w_n - l_2| = |x_{n_1} - \lambda_1| + |y_{n_1} - \mu_1| + |x_{n_2} + \lambda_2| + |y_{n_2} - \mu_2| \leq \frac{1}{4}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon + \frac{1}{4}\epsilon = \epsilon$$

$$|z_n - l_1| + |w_n - l_2| < \epsilon \Longrightarrow\ (z_n + w_n)\ \textit{is convergent.} \quad \blacksquare$$

Question 0.4 *What is meant by saying that an infinite series* $\sum_{n=1}^{+\infty} s_n$ *is convergent?*

An infinite series $\sum_{n=1}^{+\infty} s_n$ is said to converge to some $s \in \mathcal{C}$ IFF given any $\epsilon > 0\ \exists$ some $N \in \mathcal{N}$ s.t.

$$\left| \sum_{n=1}^{m} s_n - s \right| < \epsilon\ \forall\ m \geq\ N$$

Question 0.5 *Test the following infinite series for convergence.*

$$\sum_{n=1}^{+\infty} \frac{sin(n) - 4}{\sqrt{2n - 1}}$$

This diverges by the direct comparisson test.

$$\sum_{n=1}^{+\infty} \frac{7sin(n) - 4}{\sqrt{2n^3 - n^2}}$$

This converges by the direct comparisson test.

$$\sum_{n=1}^{+\infty} \frac{z^n}{(2n)!}$$

This converges by the ration test.

$$\sum_{n=1}^{+\infty} \frac{n!}{n^{n+3}}$$

This converges by the direct comparisson test.

Question 9

Question 0.1 *What is meant by saying that a sequence* $z_1, z_2, z_3, \ldots$ *of complex number is a Cauchy Sequnce.*

A sequence $z_1, z_2, z_3, \ldots$ of complex numbers is a Cauchy Sequence IFF given any $\epsilon > 0$ $\exists$ some $N \in \mathcal{N}$ s.t $|z_m - z_n| < \epsilon$ $\forall$ $m, n \geq N$.

Question 0.2 *Prove that every convergent sequence of compex number is a Cauchy Sequence.*

If the sequence $z_1, z_2, z_3, \ldots$ is convergent then $\exists$ some $N \in \mathcal{N}$ s.t given any $\epsilon > 0$ $|z_m - l| < \frac{1}{2}\epsilon$ when $m \geq N$ and $|z_n - l| < \frac{1}{2}\epsilon$ when $n \geq N$ and where $l = \lim_{n\to+\infty} z_n$

$$\implies |z_m - z_n| = |(z_m - l) - (z_n - l)| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

$\implies$ $|z_m - z_n| < \epsilon$ which means that z_n is a Caucy Sequence. ▮

Question 0.3 *Prove that every Cauchy Sequence of complex number is bounded.*

Let the sequence $z_1, z_2, z_3, \ldots$ be a Cauchy Sequence. $\implies$ $\exists$ some $N \in \mathcal{N}$ s.t. $|z_m - z_n| < 1$ $\forall$ $m, n \geq N$
In Particular $|z_n| \leq |z_N| + 1$ when $n \geq N$
$\implies$ $|z_n| \leq$ R where $R = max(|z_1|, |z_2|, \ldots, |z_{N-1}|, \ldots, |z_N| + 1)$
Therefore the sequence is bounded as required. ▮

Question 0.4 *Prove that every Cauchy sequence of complex numbers is convergent.*

Cauchy Sequences are bounded therefore have convergent subsequences by the Bolzano Weierstrass theorem.

Let $z_{n_1}, z_{n_2}, \ldots$ be one such subsequence. Therefore $\lim_{n\to+\infty} z_{n_j} = l$
We claim that $\lim_{n\to+\infty} z_n = l$
Let $\epsilon > 0$ be given therefore $\exists$ some $N \in \mathcal{N}$ s.t. $|z_m - z_n| < \frac{1}{2}\epsilon$ when $m, n \geq N$.
Let j be chosen s.t. $n_j \geq N$ and $|z_{n_j} - z_n| < \frac{1}{2}\epsilon$

$$|z_n - l| = |z_n - z_{n_j}| + |z_{n_j} - l| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \; when \; n \geq N$$

$$\implies |z_n - l| < \epsilon$$

Therefore every Cauchy Sequence is convergent. ▮

Question 10

Question 0.1 *Test the following series for convergence:*

$$\sum_{n=1}^{+\infty} \frac{z^n}{n!\sqrt{n}}$$

unsure

$$\sum_{n=1}^{+\infty} \frac{3sin(n) - 2}{n\sqrt{n}}$$

This converges by the direct comparisson test.

$$\sum_{n=2} +\infty \frac{(-1)^n}{logn}$$

This diverges by the direct comparisson test.

$$\sum_{n=1} +\infty \frac{4 + cos(n^2)}{n}$$

This diverges by the direct comparisson test.

$$\sum_{n=1} +\infty \frac{n!}{n^{n+2}}$$

This converges by the direct comparisson test.

Question 11

Question 0.1 *Prove that the infinite series* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *is divergent.*

Let s_m be the partial sum $\sum_{n=1}^{m} \frac{1}{n}$
We claim that $s_{2k} \geq \frac{k+1}{2}$ $\forall$ $k \in \mathcal{N}$.
When $k = 1$ the result is clear.
Now suppose that $s_{2k-1} \geq \frac{k}{2}$ then

$$s_{2k} = s_{2k-1} + \frac{1}{2^{k-1}+1} + \frac{1}{2^{k-1}+2} + \ldots + \ldots + \frac{1}{2^k} \geq s_{2k-1} + 2^{k-1}\left(\frac{1}{2^k}\right) = s_{k-1} + \frac{1}{2}$$

Therefore by induction on k it follows that $s_{2k} \geq \frac{k+1}{2}$
$\Longrightarrow$ the series $s_1, s_2, s_3, \ldots$ is not bounded and therefore cannot converge.
$\Longrightarrow$ the series $\sum_{n=1}^{+\infty} \frac{1}{n}$ is divergent. $\blacksquare$

Question 0.2 *Prove that the infinite series* $\sum_{n=2}^{+\infty} \frac{1}{nlogn}$

Let $a_n = \frac{1}{nlogn}$ and let $s_n = \sum_{m=2}^{n} \frac{1}{nlogn}$

$$\Longrightarrow s_{2^k} = \frac{1}{2log2} + \left(\frac{1}{3log3} + \frac{1}{4log4}\right) + \left(\frac{1}{5log5} + \frac{1}{6log6} + \frac{1}{7log7} + \frac{1}{8log8}\right)$$

$$+ \ldots + \left(\frac{1}{(2^{k-1}+1)log2^{k-1}} + \ldots + \frac{1}{2^k log2^k}\right) \geq \frac{1}{2log2} + \frac{2}{2log4} + \frac{4}{8log8}$$

$$+ \ldots + \frac{2^{k-1}}{2^k log2^k} = \frac{1}{2log2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{k}\right)$$

$\Longrightarrow$ $s_{2^k} \geq \frac{1}{2log2} \sum_{n=2}^{+\infty} \frac{1}{n}$
Therefore $s_{2^k} \to +\infty$ as $n \to +\infty$ so the series is divergent. $\blacksquare$.

Question 12

Question 0.1 *Prove that the infinite series* $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$ *is convergent.*

The Alternating Series Test states that if $a_1 > a_2 > a_3 > \ldots$ and $lim_{n \to +\infty} a_n = 0$ then the infinite series $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n$ is convergent.

Proof: Let s_m be the partial sum $\sum_{n=1}^{m} (-1)^{n-1} a_n$.

$$Now\ s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}$$

$$And\ s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k}\ \forall\ k \in \mathcal{N}$$

$\Longrightarrow$ the subsequence $s_1, s_3, s_5, \ldots$ is non-increasing.
$\Longrightarrow$ the subsequence $s_2, s_4, s_6, \ldots$ is non-decreasing.

$$But\ s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$$

$\Longrightarrow$ These subsequences are bounded and convergent. (Every non-increasing or decreasing bounded sequence is convergent), and

$$\lim_{n \to +\infty} s_{2k} = \lim_{n \to +\infty} s_{2k-1} = s$$

We claim that $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n = s$

Let $\epsilon > 0$ be given. Then $\exists\ K_1, K_2 \in\ \mathcal{N}$ s.t.

$$|s - s_{2k-1}| < \epsilon\ when\ k\ \geq K_1$$

$$and\ |s - s_{2k}| < \epsilon\ when\ k\ \geq K_2$$

Choose N s.t. $N \geq$ ”$K_1 - 1$ and $N \geq 2K_2$

$$\Longrightarrow |s - s_m| < \epsilon\ when\ m\ \geq N$$

$$\Longrightarrow \sum_{n=1}^{+\infty} (-1)^{n-1} a_n = \lim_{n \to +\infty} s_m = s$$

Therefore is convergent. ▮

Using this result and since in the case of $\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n}$ where $a_n > a_{n-1}\ \forall\ n$ and $\lim_{n \to +\infty} a_n = 0$ this series converge.

1 Q

uestion 13

Question 1.1 *State the Alternating Series Test and prove that any infinite series satisfying its conditions is convergent.*

The Alternating Series Test states that if $a_1 > a_2 > a_3 > \ldots$ and $lim_{n \to +\infty} a_n = 0$ then the infinite series $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n$ is convergent.

Proof: Let s_m be the partial sum $\sum_{n=1}^{m} (-1)^{n-1} a_n$.

$$Now\ s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}$$

$$And\ s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k}\ \forall\ k \in \mathcal{N}$$

$\Longrightarrow$ the subsequence $s_1, s_3, s_5, \ldots$ is non-increasing.
$\Longrightarrow$ the subsequence $s_2, s_4, s_6, \ldots$ is non-decreasing.

$$But\ s_2 \leq s_{2k} \leq s_{2k-1} \leq s_1$$

$\Longrightarrow$ These subsequences are bounded and convergent. (Every non-increasing or decreasing bounded sequence is convergent), and

$$\lim_{n \to +\infty} s_{2k} = \lim_{n \to +\infty} s_{2k-1} = s$$

We claim that $\sum_{n=1}^{+\infty} (-1)^{n-1} a_n = s$

Let $\epsilon > 0$ be given. Then $\exists\ K_1, K_2 \in\ \mathcal{N}$ s.t.

$$|s - s_{2k-1}| < \epsilon\ when\ k\ \geq K_1$$

$$and\ |s - s_{2k}| < \epsilon\ when\ k\ \geq K_2$$

Choose N s.t. $N \geq {}"K_1 - 1$ and $N \geq 2K_2$

$$\Longrightarrow |s - s_m| < \epsilon\ when\ m\ \geq N$$

$$\Longrightarrow \sum_{n=1}^{+\infty} (-1)^{n-1} a_n = \lim_{n \to +\infty} s_m = s$$

Therefore is convergent. ∎

Question 1.2 *Does the infinite series*

$$\sum_{n=1}^{+\infty} \frac{cos n\pi + sin \frac{1}{2} n\pi}{n^2}$$

satisfy the conditions of the Alternating Series Test?

No it does not satisfy the conditions of the Alternating Series Test. But it does converge using the Direct Comparisson Test.

Question 14

Question 0.1 *Determine which of the following subsets of $\mathcal{R}^2$ are open and which are closed:*

$$\{f(x,y) \in \mathcal{R}^2 : (x-2)^2 + y^2 < 9\}$$

This subset is open, because firstly it does't contain its limit points, secondly because its complement $\mathcal{R}^2 \backslash f(x,y)$ is closed.

$$\{f(x,y) \in \mathcal{R}^2 : (x-2)^2 + y^2 \geq 9\}$$

This subset is closed, because it contains its limit points and its complement is open.

$$\{f(x,y) \in \mathcal{R}^2 : (x-2)^2 + y^2 < 9 \text{ } and \text{ } x \leq \ 0\}$$

This subset is neither open or closed. This is because the $(x-2)^2 + y^2 < 9$ would make the set open but the $x \geq \ 0$ would make it closed.

$$\{f(x,y) \in \mathcal{R}^2 : (x-2)^2 + y^2 \geq 9 \text{ } or \text{ } x \leq \ 0\}$$

This subset is closed because both sections of the set. ie the $(x-2)^2+y^2 \geq 9$ part is closed and the $x \geq \ 0$ would also make the subset closed.

Question 16

Question 0.1 *Prove that a sequence* $x_1, x_2, x_3 \ldots$ *of points of* $\mathcal{R}^2$ *converges to some point* p *IFF, given any open set* U *which contains* p, $\exists$ *some* $N \in \mathcal{N}$ *s.t. the point* x_j *belongs to* $U \forall\ j$ *s.t.* $j \geq N$.

Suppose that the sequence $x_1, x_2, x_3 \ldots$ has the property that given any open set U containing p *exists* some $N \in \mathcal{N}$ s.t. $x_j \ \in \ U$ when $j \geq N$.

Let $\epsilon > 0$ be given. The open ball $B(p, \epsilon)$ is an open set. $\implies \ \exists$ some $N \in \mathcal{N}$ s.t. $x_j \in B(p, epailon)$ when $j \geq N$.
$\implies |x_j - p| < \epsilon$ when $j \geq N$. Therefore the sequence converges.

Conversely. Suppose that the sequence converges to some p. Let U be an open set containing p. Then $\exists$ some $\epsilon > 0$ s.t. the open ball $B(p, \epsilon)$ of radius p arround ϵ is a subset of U.

$\implies \ \exists$ some $\epsilon > 0$ s.t. U contains all points $x \ of \ X$ s.t. $|x - p| < \epsilon$. But $\exists$ some $Nin \ \mathcal{N}$ s.t. $|x_j - p| < \epsilon$ when $j \geq N$ since the sequence converges to p.
$\implies \ x_j \in U$ when $j \geq N$ ■.

Question 0.2 *Using part (a) or otherwise show that if* F *is a closed set in* $\mathcal{R}^2$ *and if* $x_1, x_2, x_3, \ldots$ *is an infinite series of points of* $\mathcal{R}^2$ *belonging to* F *which converges to some complex number* p *then* $p \in F$.

The complement of $F, \mathcal{R}^2 \backslash F$ is open bye definition. $\implies$ Suppose $p \in \mathcal{R}^2 \backslash F$ then it follows that $x_j \in \mathcal{R}^2 \backslash F \ \forall \ j \geq N$.

This contradicts the fact that $x_j \in F \ \forall \ j \implies p \in \ F$ as required. ■