

# *Maths and Wireless Networking*

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## *The Plan*

- Look at some mathematics used in wireless networks.
- Fourier Analysis and Radio Waves.
- Probability and the WiFi MAC Layer.
- Graph Colouring and Channel Assignment.

# Wireless Information

1. Want to send some info.
2. We use a signal to do it.
3. Can just be a function  $f(t)$ .
4. E.g. take a sound wave ...
5. ... resend as radio wave.

## Changing Frequency

What to send  $f(t)$  but at different frequency.  
Use Fourier Analysis:

$$f(t) = \int c(\omega) \cos(\omega t) + s(\omega) \sin(\omega t) d\omega$$

Possible for a wide class of functions.  
For simplicity assume:

$$f(t) = \int s(\omega) \sin(\omega t) d\omega$$

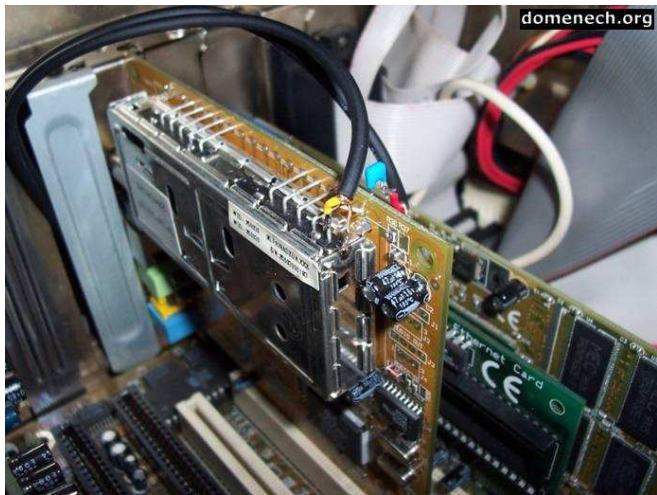
If we want to change frequency, multiply by  $\sin \omega_0 t$ .

$$\begin{aligned}\sin(\omega_0 t)f(t) &= \sin(\omega_0 t) \int s(\omega) \sin(\omega t) d\omega \\ &= \int s(\omega) \sin(\omega_0 t) \sin(\omega t) d\omega \\ &= \int \frac{s(\omega)}{2} \sin((\omega_0 - \omega)t + 90) d\omega \\ &\quad - \int \frac{s(\omega)}{2} \sin((\omega_0 + \omega)t + 90) d\omega\end{aligned}$$

Two copies of signal on either side of  $\omega_0$ .

# *Software Radio*

- People used to build circuits to do this.
- Now you can write code to do it.
- Means you can have more flexible modulation.
- You can even do it in your back yard.



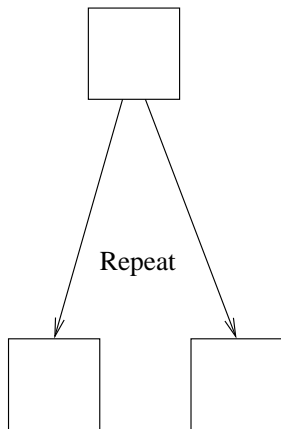
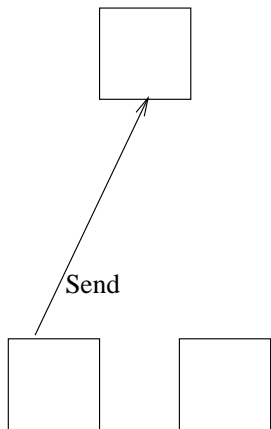




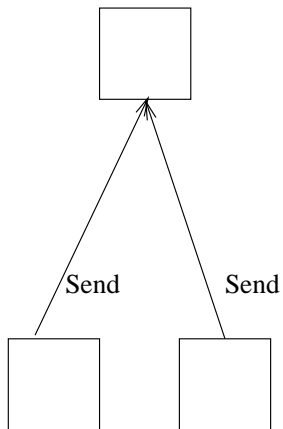
# *ALOHA*

- MAC protocol.
- Used in University of Hawaii (1970).
- Basic idea: transmit and retry if failed.
- Idea has been reused again and again.

## *Good Transmission*



# Collision

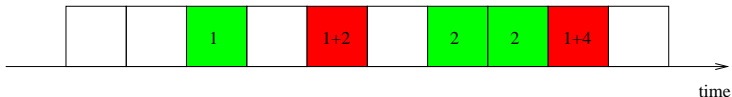


**\*silence\***



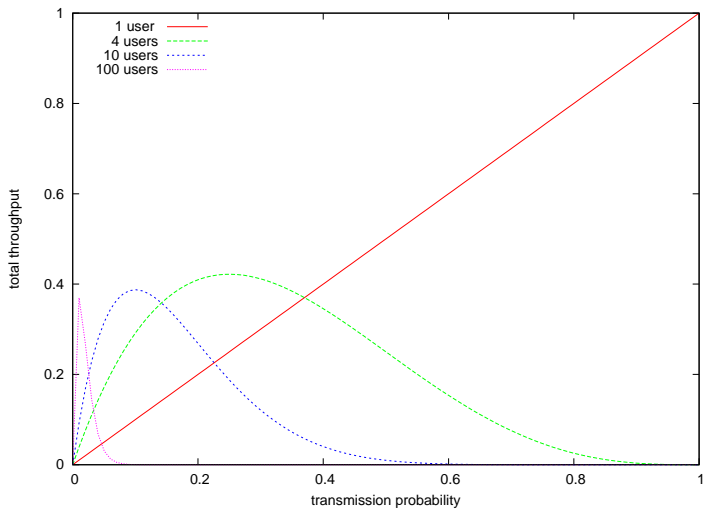
# Slotted Aloha

If you have data, transmit with probability  $\tau$ .



How can we model this?

no transmission	$(1 - \tau)^n$
good transmission	$n\tau(1 - \tau)^{n-1}$
collision transmission	$\sum_{r=2}^n \binom{n}{r} \tau^r (1 - \tau)^{n-r}$



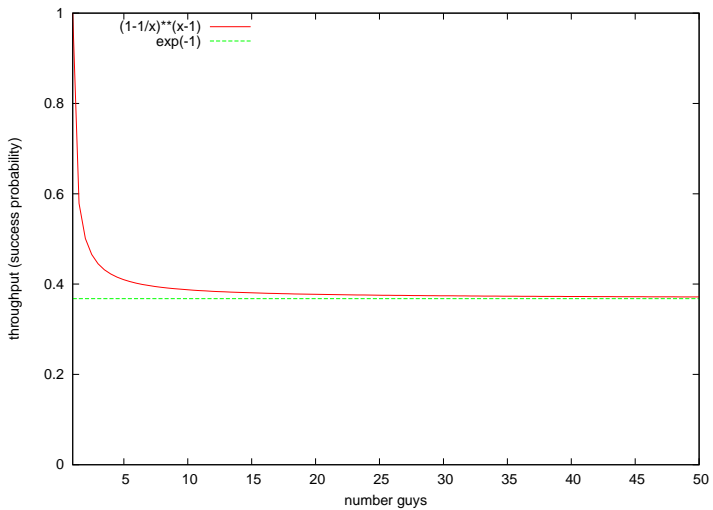
## Some Analysis

How should we pick  $\tau$ ?

$$\frac{d}{d\tau} [n\tau(1 - \tau)^{n-1}] = (1 - \tau n)(1 - \tau)^{n-2}$$

Best throughput at  $\tau = 1/n$  is  $(1 - 1/n)^{n-1}$ .

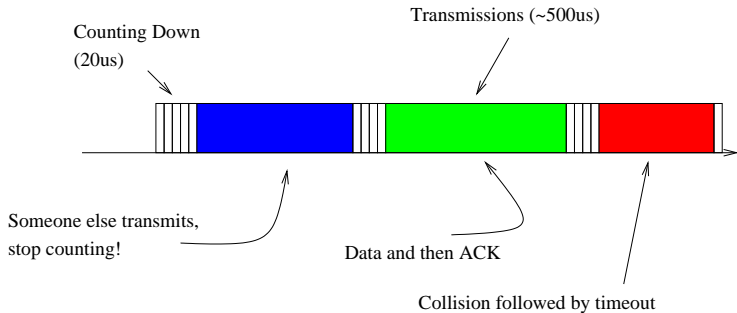
Asymptotic throughput  $1/e$ !



## What About WiFi

- After transmission choose  $\text{rand}(0, CW-1)$ .
- Wait until medium idle.
- Count down in slots.
- Transmit when get to 0.
- If ACK then  $CW \leftarrow CW_{min}$  else  $CW \leftarrow 2CW$ .





Like ALOHA like: slots different size; How to find  $\tau$ ?

## Markov Chain

Structure in probability: next state depends only on present.

E.g. If it is dry today, it will be dry tomorrow with prob 0.6.  
If it rains today, it rains tomorrow with prob 0.8.

$$\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

This is called transition matrix.

$$A^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}, \quad A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.56 \end{pmatrix},$$

$$A^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.376 \\ 0.624 \end{pmatrix}, \quad A^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3504 \\ 0.6496 \end{pmatrix},$$

$$A^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.34016 \\ 0.65984 \end{pmatrix}, \dots \quad A^{600} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.33333 \\ 0.66666 \end{pmatrix}.$$

Eigenvectors are  $\begin{pmatrix} 1 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$  and  $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ .

# *Applications of Markov Chains*

- Queueing theory,
- Physics,
- Economics,
- Google,
- Spammers.

*No state* all and of fierce to a rods no gathered in and will of kingdoms turneth lay as the answered lamp not bullocks, flax, egypt. because the sleep, the him mine top their son mother, plenty one planted the but comfort he king, be possessed whom among there. mother shall displeased that and of and thou the of eighth to and him, thou you; they and him, the because horns man stablish enquire so i all and do the about. in knew can but mercy the do shall the that why their them. which beloved be field, thence shall and my

*1 word* the other on a truth, nor your god, that also is come, they were more with assyria, and fro through the thunder they wax at liberty of the night and departed, he said, god do i have. and to give them that holdest mine own pleasure, nor to caesar, or sojourner among you; and shall swear unto the tenth part it was gathered together of the lord shall live will slay them. wherefore then spake very good. know them that after his thoughts. for and he shall sit, one of all the sake, o thou art and his god hath

*2 words* died, so shall ye lodge, o ye shepherds, and cry; and came to jesus, they besought him that dwelleth in this and i delivered the commissions unto the levite, he hath redeemed my soul made me to bring them before them; and ye know whence i came forth of me, saith the lord god; behold, i have commanded you; and ye shall eat unclean things in your behold my hands; and said, o my god, to do evil; learn to fear the lord, from the earth, and let him alone. their drink offerings; and as they were over the then

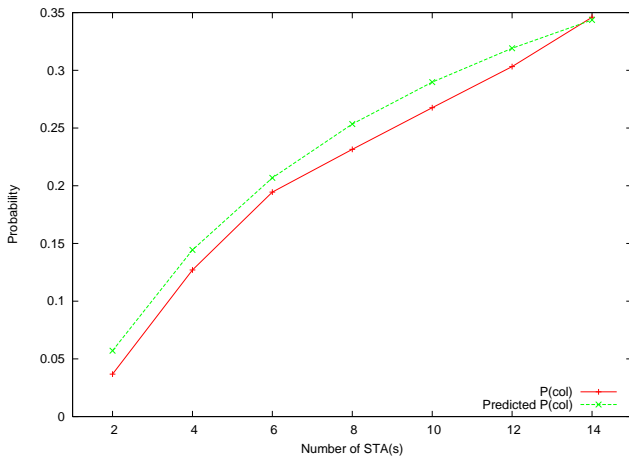
*3 words* himself had silver, and gold, and make crowns, and set them in their synagogue, insomuch that they pressed upon him for to touch him, as many as obeyed him, were dispersed. and now i am no more worthy to be beaten, that the judge shall cause him to fall upon it suddenly, and terrors upon the city. she that hath many children is waxed feeble. the lord killeth, and maketh he bringeth low, and lifteth up. he raiseth up himself, the mighty are gathered against me; not for my transgression, the fruit of thy works. he causeth the vapours to

## *Bianchi: Mean Field*

1. Assume fixed chance of collision  $p$ .
2. Use CW and counter as state.
3. Build Markov Chain  $A(p)$  based on 802.11.
4. Find stationary distribution  $\vec{b}$ .
5. Add up transmitting states to get  $\tau$ .

Now we have  $\tau(p)$ , but also  $1 - p = (1 - \tau)^{n-1}$ .

# How Good Is This?





## *The other direction*

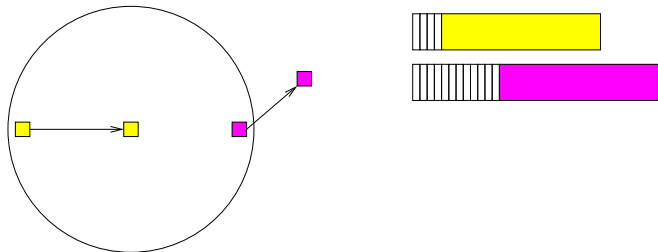
- Bianchi assumes fixed  $p$ .
- State space from  $\approx 2000^n$  to 2000.
- Gives exact solution.
- Why does this give good results?
- Turns out  $p$  is close to fixed, but not IID.
- Not yet understood, question in probability/linear algebra.

## WiFi Channels

- 802.11b/g sends signals at about 2.4Ghz.
- In fact, 11 (or 13/14) channels.
- But signals are a few channels wide!
- Channels 1, 6 and 11 are orthogonal.
- What happens if networks overlap?

## Hidden Node

If everyone can hear, good. Otherwise...

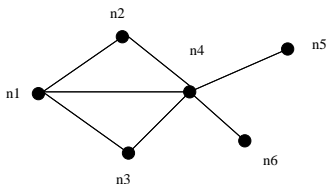


Can result in pretty bad performance.

# Graphs

A graph is a handy structure with vertices  $V$  and edges  $E \subset V^2$ .

1.  $(n, n)$  is never an edge.
2.  $(n_1, n_2) \in E \Rightarrow (n_2, n_1) \in E$ .

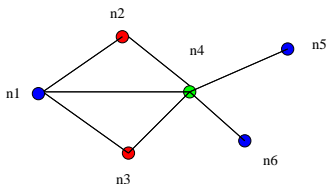


$$E = \{(n_1, n_2), (n_1, n_3), (n_1, n_4), (n_2, n_4), (n_3, n_4), (n_4, n_5), (n_4, n_6)\}.$$

# Graph Colouring

A node colouring of a graph is a function  $c$  from the vertices  $V$  to a set of colours  $C$ .

1. Sometimes no restriction on  $c$ .
2. Sometimes want  $c(n_1) \neq c(n_2)$  if  $(n_1, n_2) \in E$ .



$$c(n_1) = B, c(n_2) = R, c(n_3) = R, c(n_4) = G, c(n_5) = B, \dots$$

## *Picking a Channel*

- Channel assignment is graph colouring.
- Know the graph? Use centralised scheme.
- Distributed schemes exist too.
- What about when stations can't talk?

## Colouring in the Dark

Keep changing colour until no clash.

1. Initially  $\vec{p} \leftarrow (1/C, \dots, 1/C)$ .
2. Choose a channel according to  $\vec{p}$ .
3. No Clash?  $\vec{p} \leftarrow (0, 0, \dots, 1, \dots)$ .
4. Clash?  $\vec{p} \leftarrow (1 - \beta)\vec{p} + \frac{\beta}{C-1}(1, 1, \dots, 0, \dots)$ .
5. Goto 2.

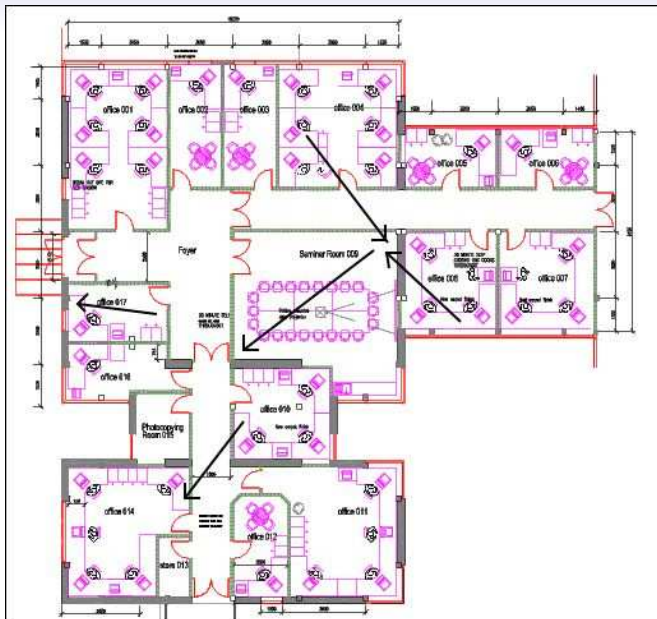
## How Fast?

Unfinished after  $T$  steps with probability better than:

$$\left(1 - A \left( (1 - \beta)^2 \left( \frac{\beta}{C-1} \right)^{2C+1} \right)^N \right)^{\frac{T}{N}}$$

Where  $A$  is the number of good colourings,  $N$  is number of nodes,  $C$  is number of colours.





## *Wrapping Up*

- Used Fourier Analysis for radio.
- Used Markov chains for predicting WiFi.
- Used Graph theory for channel allocation.
- Also lots of mathematics at other layers!

`http://www.hamilton.ie/`