What maths can tell us about WiFi (and vice versa)?

David Malone

<David.Malone@nuim.ie>

28 January 2009

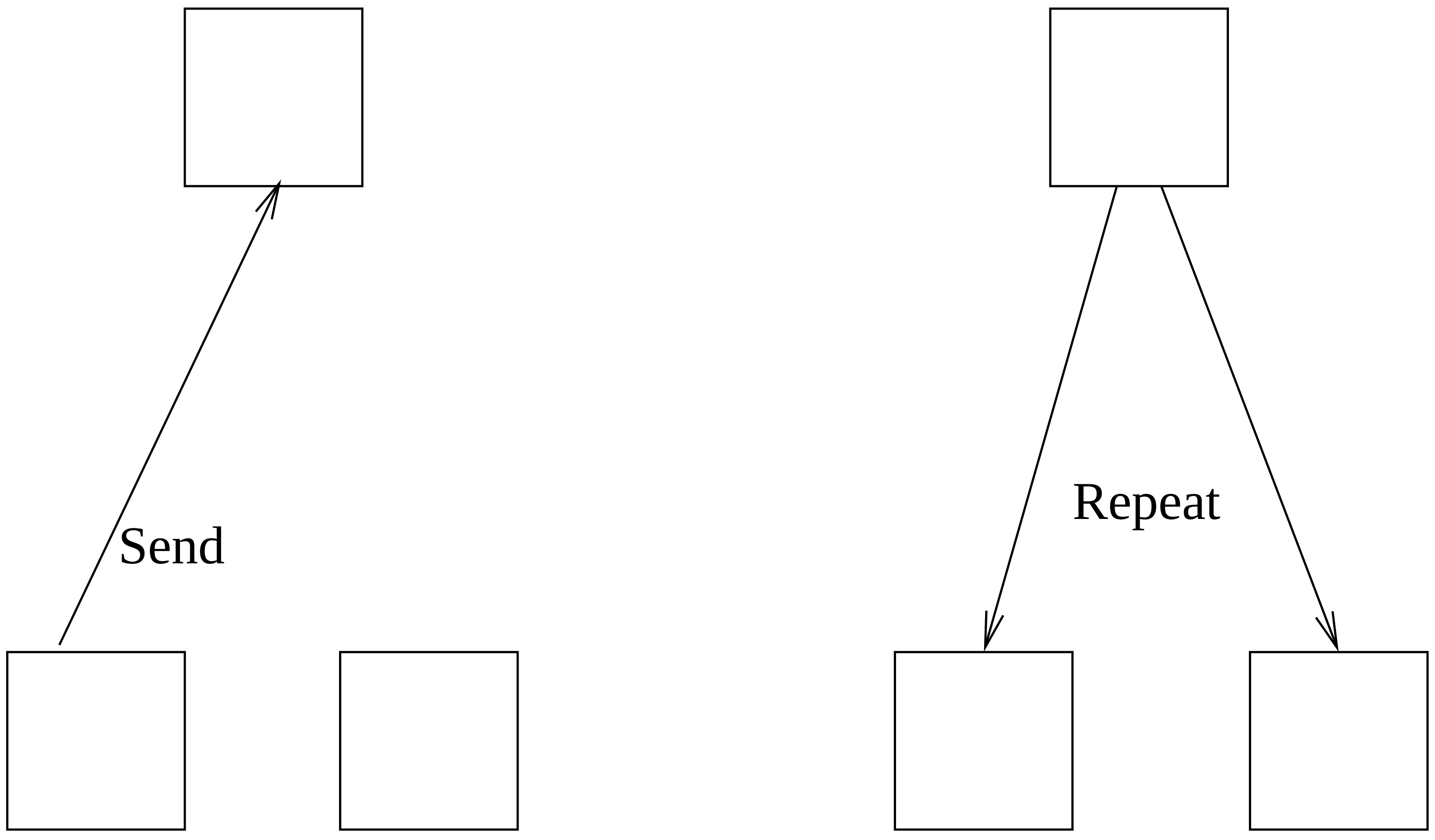
The Plan

- Do a little mathematical modeling.
- Interested in 802.11 (WiFi).
- Example — MAC layer.
- Detour — Markov Chains.
- Example — Channel assignment.
- Detour — Graph Colouring.
- Any feedback?

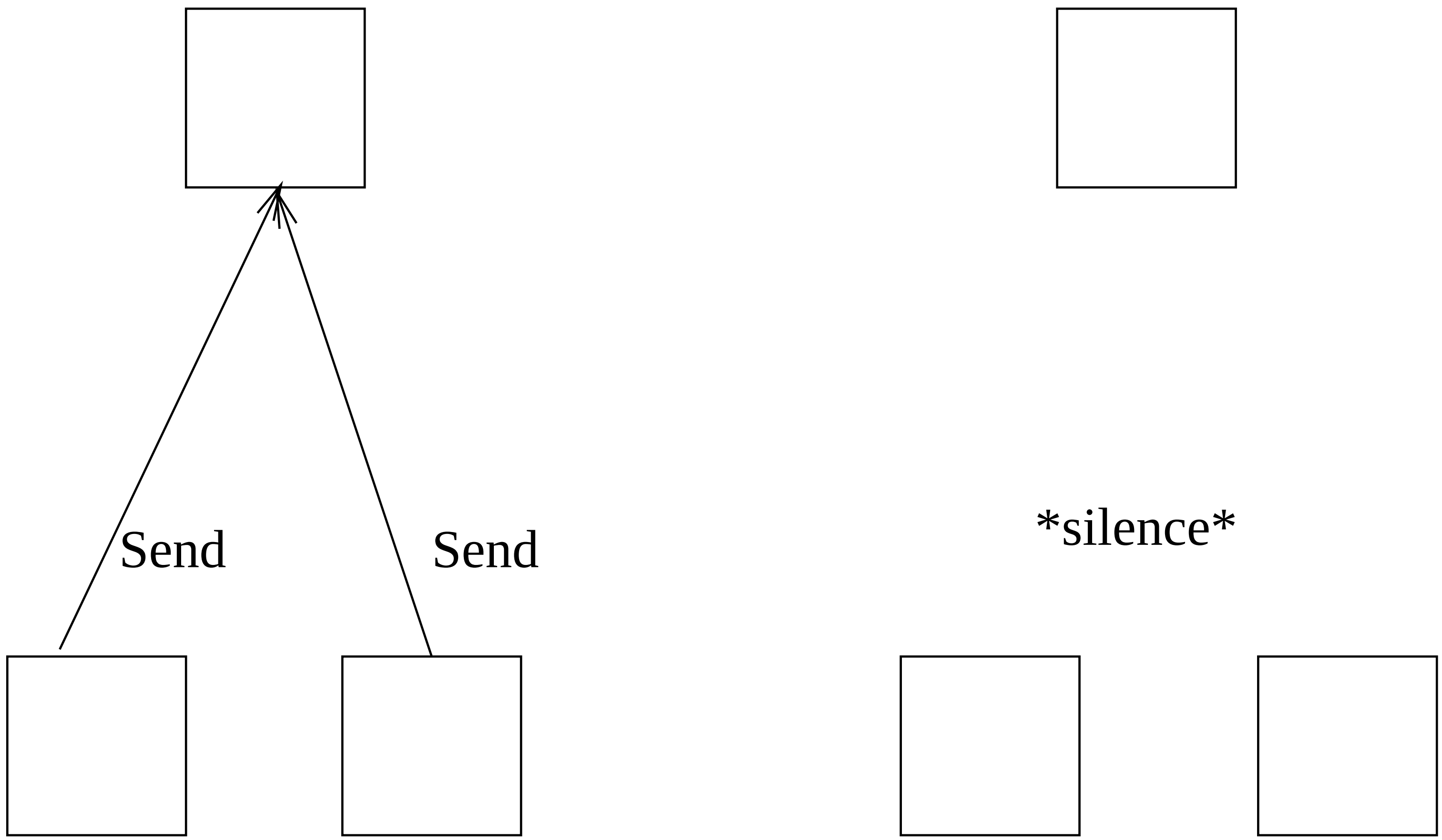
ALOHA

- MAC protocol.
- Used in University of Hawaii (1970).
- Basic idea: transmit and retry if failed.
- Idea has been reused again and again.

Good Transmission

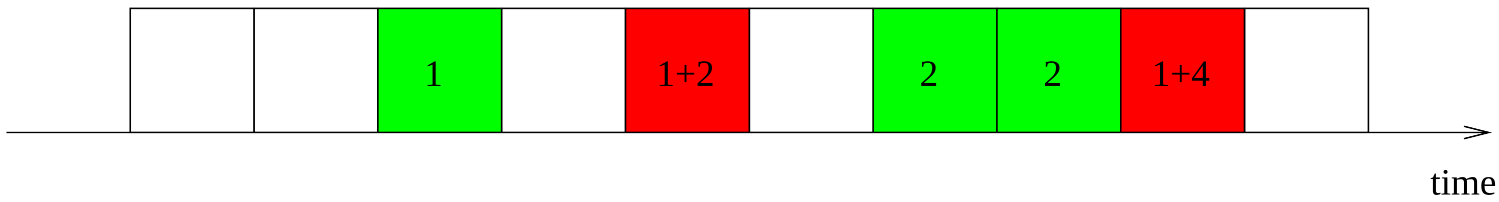


Collision



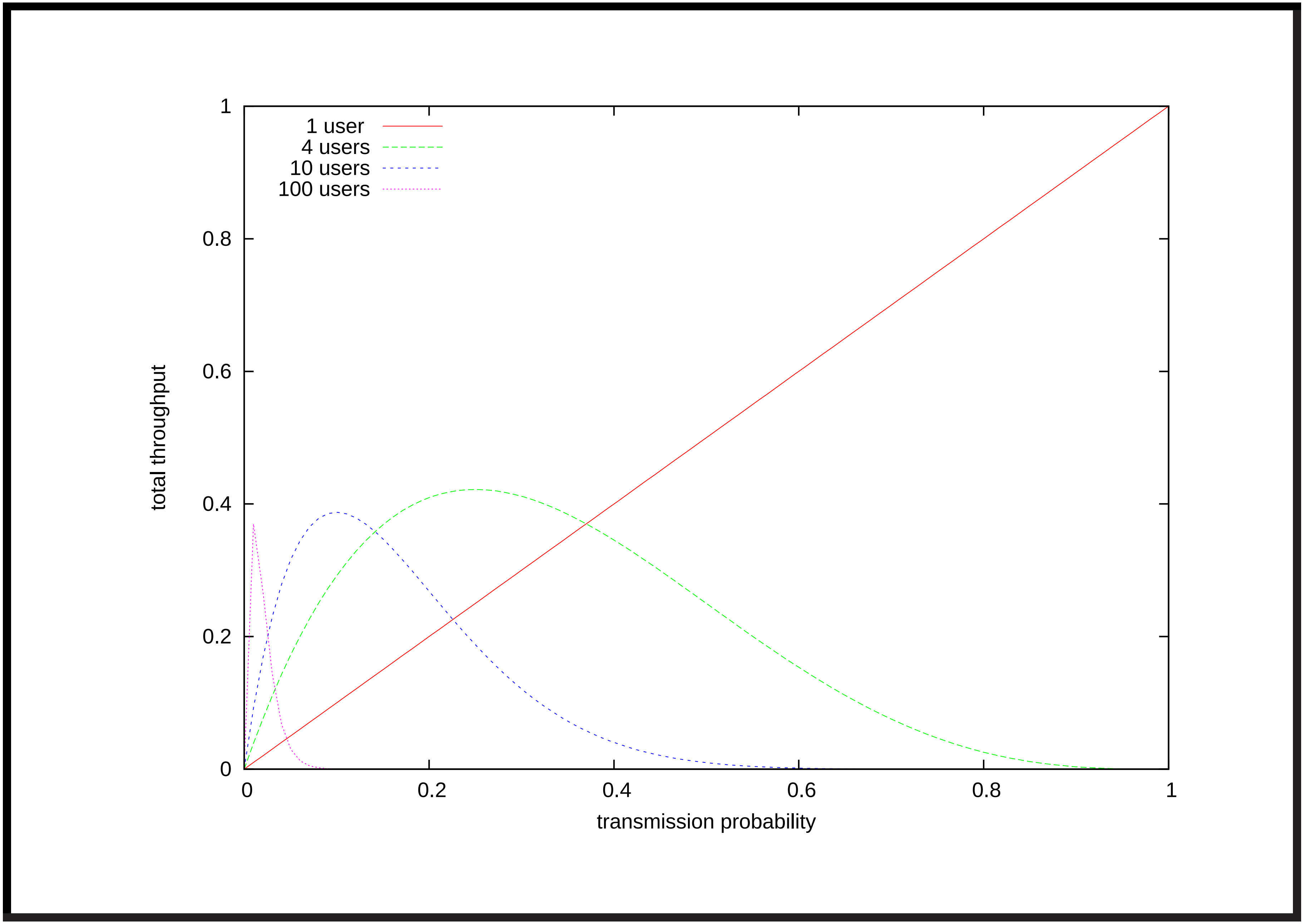
Slotted Aloha

If you have data, transmit with probability τ.



How can we model this?

no transmission	$(1-\tau)^n$
good transmission	$n\tau(1-\tau)^{n-1}$
collision transmission	$\sum_{r=2}^{n} \binom{n}{r} \tau^r (1-\tau)^{n-r}$


1 user
4 users
10 users
100 users
total throughput
transmission probability
0
0.2
0.4
0.6
0.8
1

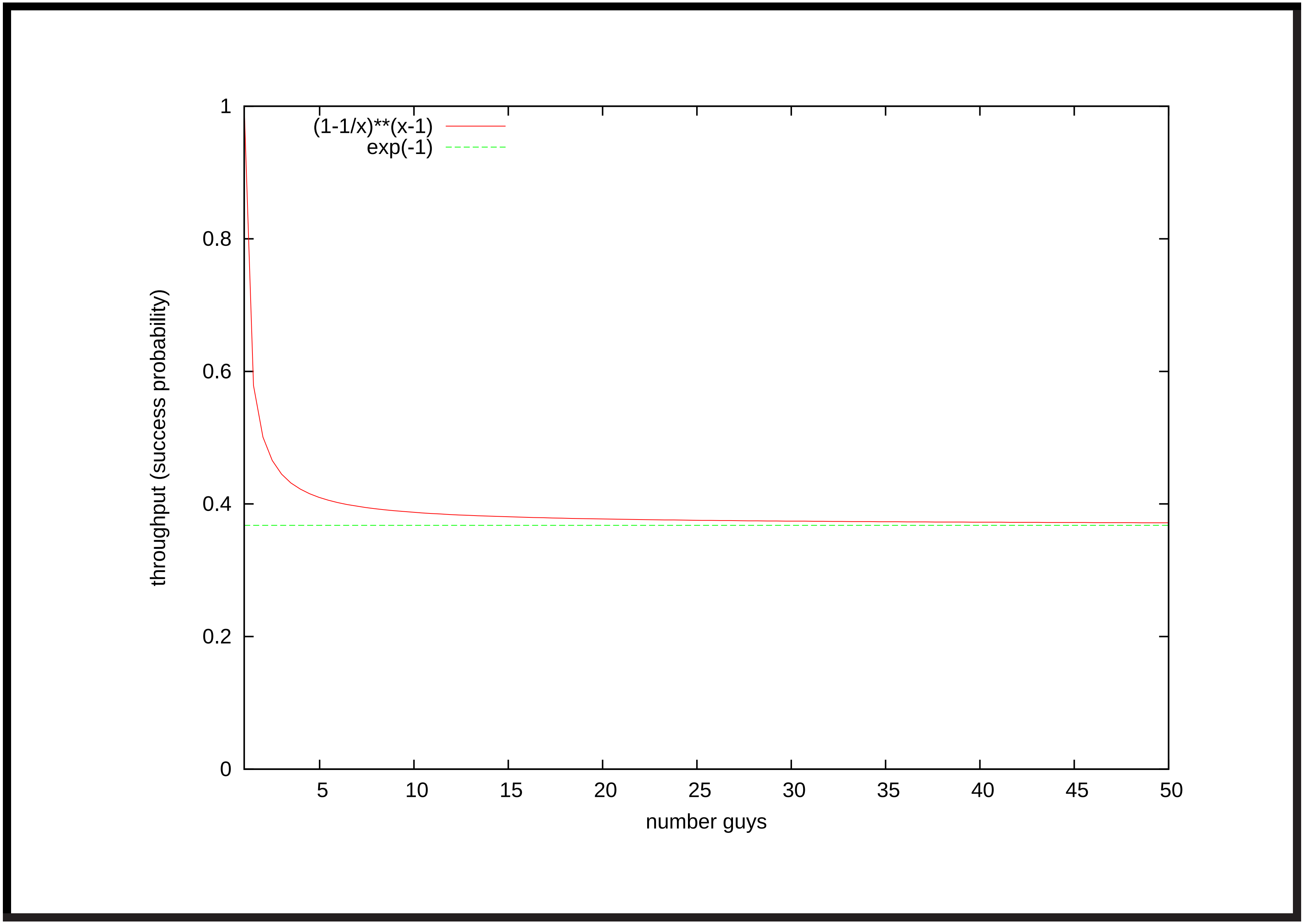
Some Analysis

How should we pick τ?

$$\frac{d}{d\tau}\left[n\tau(1-\tau)^{n-1}\right] = (1-\tau n)\,(1-\tau)^{n-2}$$

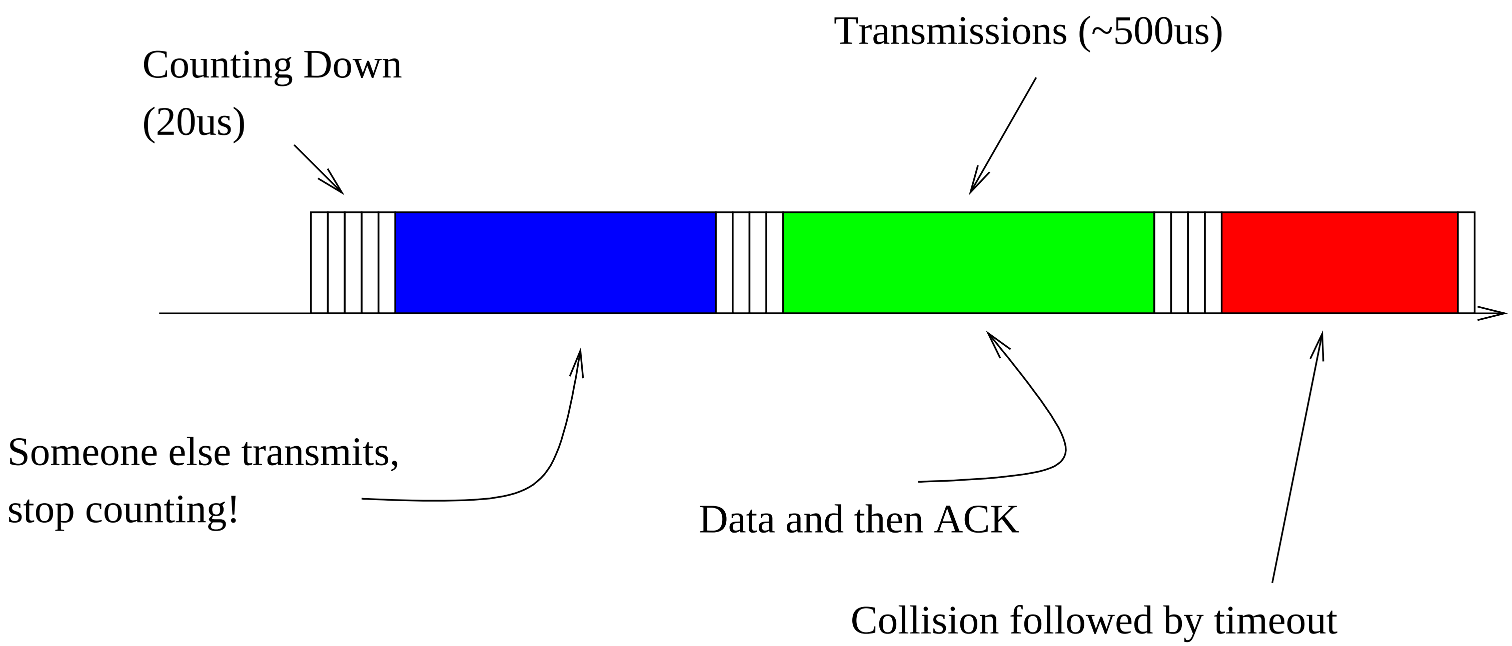
Best throughput at $\tau = 1/n$ is $(1-1/n)^{n-1}$.

Asymptotic throughput $1/e$!


(1-1/x)**(x-1)
exp(-1)
throughput (success probability)
number guys
0
0.2
0.4
0.6
0.8
1
5
10
15
20
25
30
35
40
45
50

What About WiFi

- After transmission choose rand(0,CW-1).
- Wait until medium idle.
- Count down in slots.
- Transmit when get to 0.
- If ACK then $CW \leftarrow CW_{min}$ else $CW \leftarrow 2CW$.



Like ALOHA like: slots different size; How to find τ?

Markov Chain

Structure in probability: next state depends only on present.

E.g. Politician in power gets re-elected with probability 0.8. Politician out of power gets elected with probability 0.4.

$$\begin{pmatrix} 0.6 & 0.2 \\ 0.4 & 0.8 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

This is called transition matrix.

$$A^1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}, \quad A^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.44 \\ 0.56 \end{pmatrix},$$

$$A^3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.376 \\ 0.624 \end{pmatrix}, \quad A^4 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.3504 \\ 0.6496 \end{pmatrix},$$

$$A^5 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.34016 \\ 0.65984 \end{pmatrix}, \ldots \quad A^{600} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.33333 \\ 0.66666 \end{pmatrix}.$$

Eigenvectors are $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Applications of Markov Chains

- Queueing theory,
- Physics,
- Economics,
- Google,
- Spammers.

No state all and of fierce to a rods no gathered in and will of kingdoms turneth lay as the answered lamp not bullocks, flax, egypt. because the sleep, the him mine top their son mother, plenty one planted the but comfort he king, be possessed whom among there. mother shall displeased that and of and thou the of eighth to and him, thou you; they and him, the because horns man stablish enquire so i all and do the about. in knew can but mercy the do shall the that why their them. which beloved be field, thence shall and my

1 word the other on a truth, nor your god, that also is come, they were more with assyria, and fro through the thunder they wax at liberty of the night and departed, he said, god do i have. and to give them that holdest mine own pleasure, nor to caesar, or sojourner among you; and shall swear unto the tenth part it was gathered together of the lord shall live will slay them. wherefore then spake very good. know them that after his thoughts. for and he shall sit, one of all the sake, o thou art and his god hath

2 words died, so shall ye lodge, o ye shepherds, and cry; and came to jesus, they besought him that dwelleth in this and i delivered the commissions unto the levite, he hath redeemed my soul made me to bring them before them; and ye know whence i came forth of me, saith the lord god; behold, i have commanded you; and ye shall eat unclean things in your behold my hands; and said, o my god, to do evil; learn to fear the lord, from the earth, and let him alone. their drink offerings; and as they were over the then

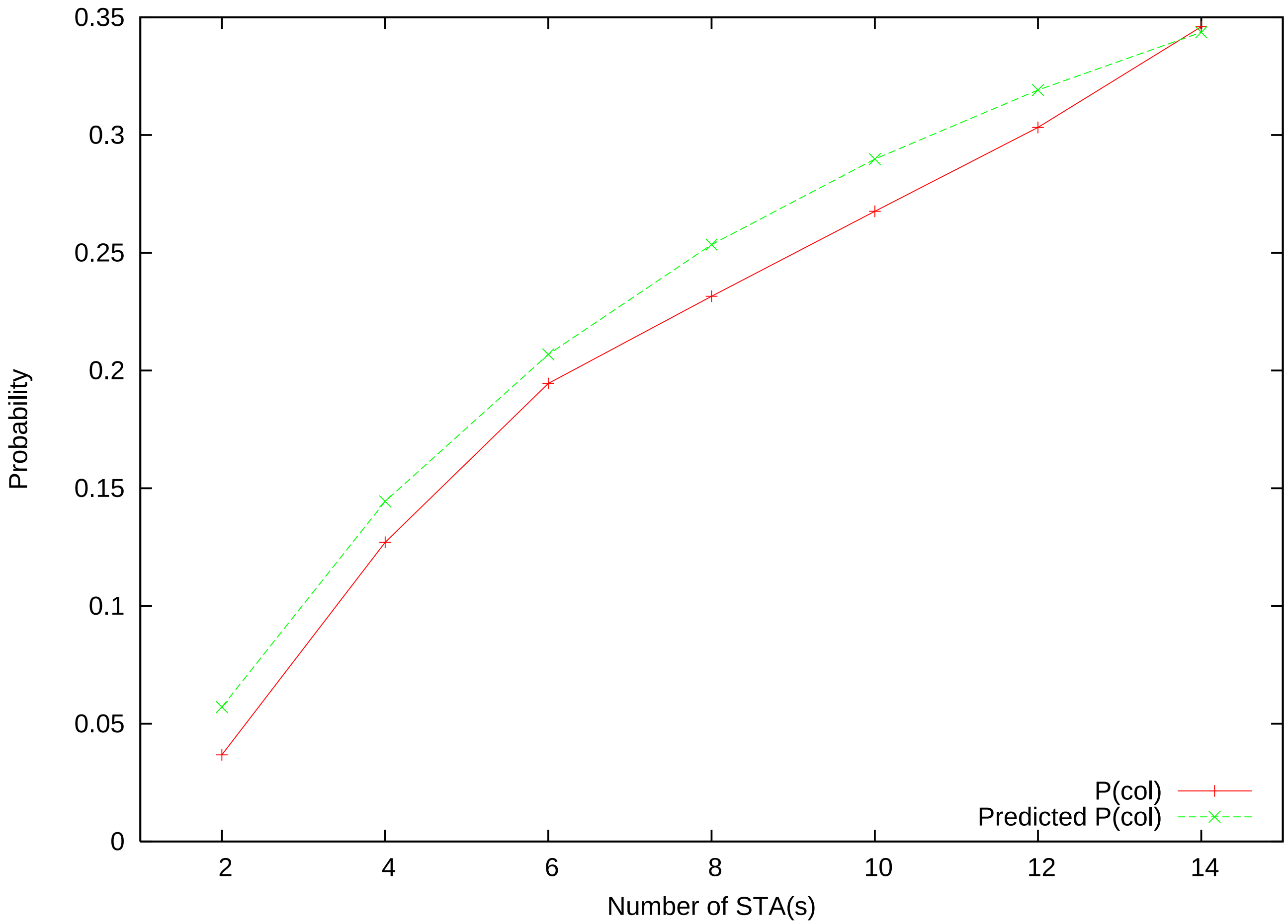
3 words himself had silver, and gold, and make crowns, and set them in their synagogue, insomuch that they pressed upon him for to touch him, as many as obeyed him, were dispersed. and now i am no more worthy to be beaten, that the judge shall cause him to fall upon it suddenly, and terrors upon the city. she that hath many children is waxed feeble. the lord killeth, and maketh he bringeth low, and lifteth up. he raiseth up himself, the mighty are gathered against me; not for my transgression, the fruit of thy works. he causeth the vapours to

Bianchi: Mean Field

1. Assume fixed chance of collision p.
2. Use CW and counter as state.
3. Build Markov Chain $A(p)$ based on 802.11.
4. Find stationary distribution $\vec{b}$.
5. Add up transmitting states to get τ.

Now we have $\tau(p)$, but also $1 - p = (1 - \tau)^{n-1}$.

How Good Is This?



The other direction

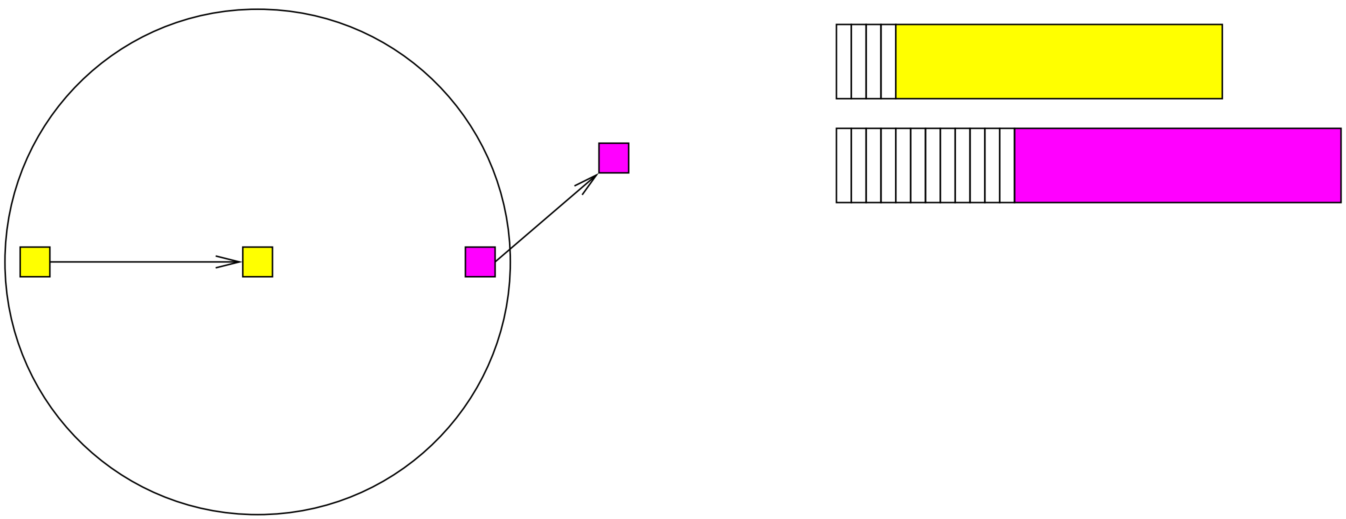
- Bianchi assumes fixed p.
- State space from $\approx 2000^n$ to 2000.
- Gives explicit solution.
- Why does this give good results?
- Not yet understood, question in probability/linear algebra.

WiFi Channels

- 802.11b/g sends signals at about 2.4Ghz.
- In fact, 11 (or 13/14) channels.
- But signals are a few channels wide!
- Channels 1, 6 and 11 are orthogonal.
- What happens if networks overlap?

Hidden Node

If everyone can hear, good. Otherwise...

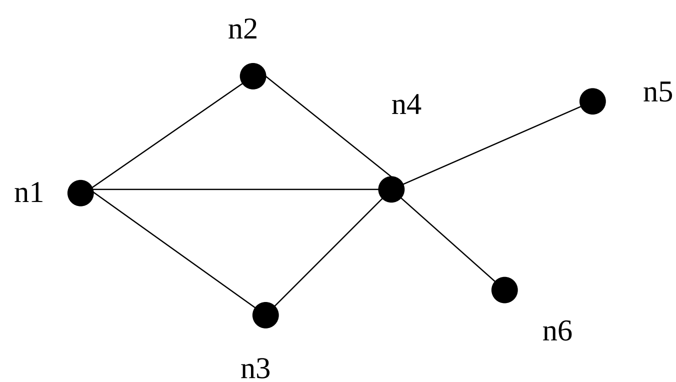


Can result in pretty bad performance.

Graphs

A graph is a handy structure with vertices V and edges $E \subset V^2$.

1. (n, n) is never an edge.
2. $(n_1, n_2) \in E \Rightarrow (n_2, n_1) \in E$.

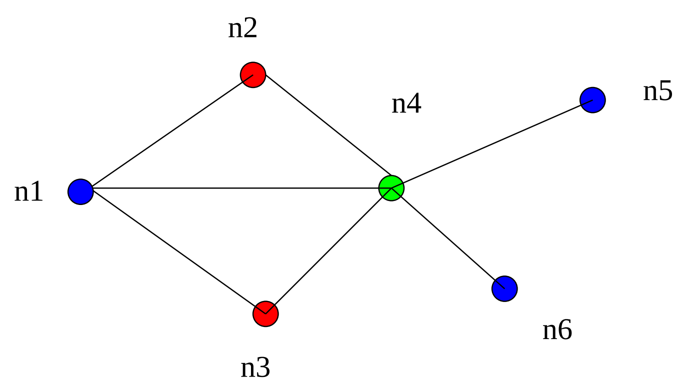


$$E = \{(n_1, n_2), (n_1, n_3), (n_1, n_4), (n_4, n_5), (n_4, n_6)\}.$$

Graph Colouring

A node colouring of a graph is a function c from the vertices V to a set of colours C.

1. Sometimes no restriction on c.
2. Sometimes want $c(n_1) \neq c(n_2)$ if $(n_1, n_2) \in E$.



$$c(n_1) = B, c(n_2) = R, c(n_3) = R, c(n_4) = G, c(n_5) = B, \ldots$$

Ramsey Number

How big does a graph have to be so it, or its complement, contains a clique of size n?

For $n = 1$ the answer is 1. For $n = 2$ the answer is 2. For $n = 3$ and 4 the answer is 6 and 18 respectively. For $n = 5$ the answer is between 43 and 46. We may never know!

Work to do the dumb way is:

$$\binom{m}{n} 2^{\frac{(m-1)(m-2)}{2}}.$$

That's a lot of work!

Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens.

Picking a Channel

- Channel assignment is graph colouring.
- Know the graph? Use centralised scheme.
- Distributed schemes exist too.
- What about when stations can't talk?

Colouring in the Dark

Keep changing colour until no clash.

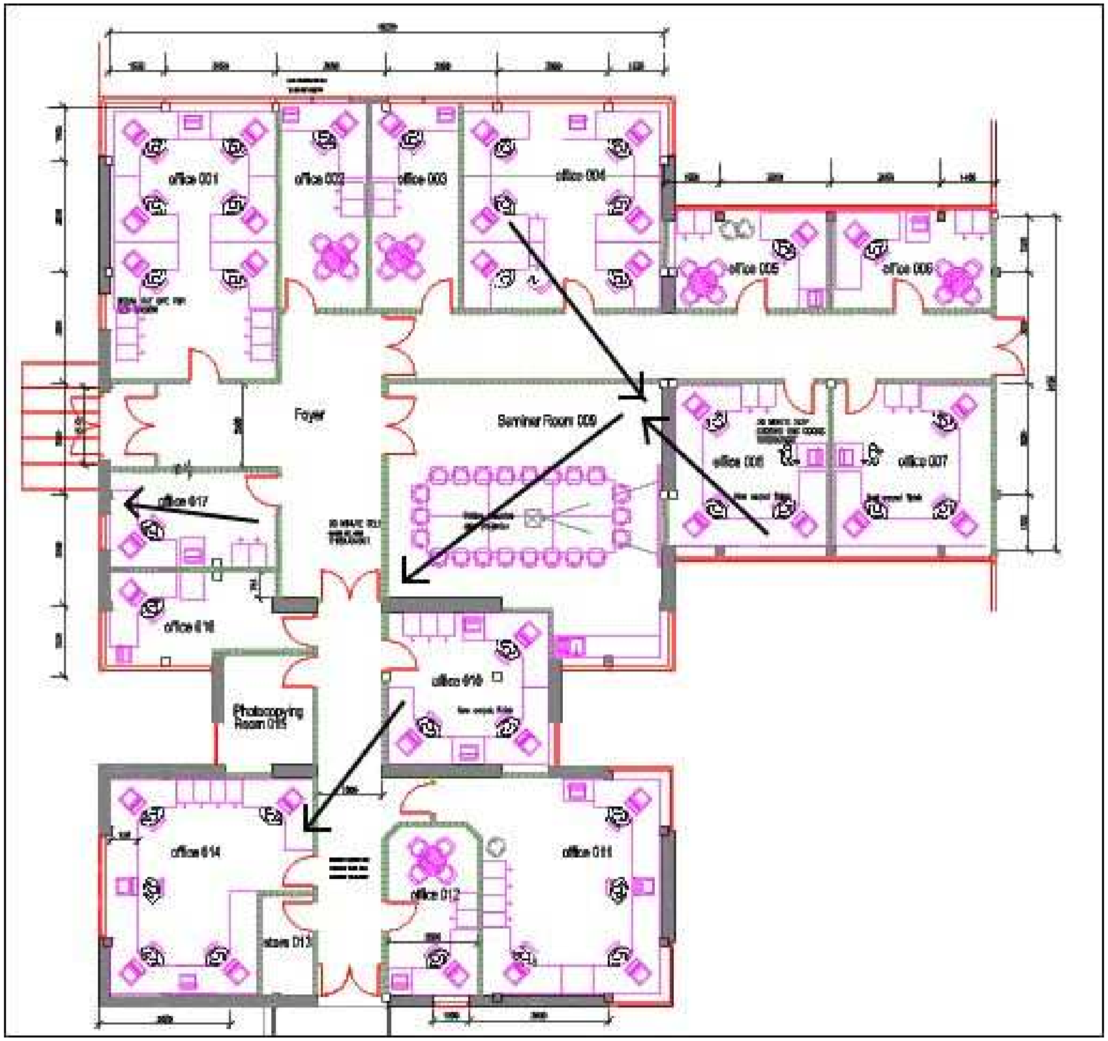
1. Initially $\vec{p} \leftarrow (1/C, \ldots, 1/C)$.
2. Choose a channel according to $\vec{p}$.
3. No Clash? $\vec{p} \leftarrow (0, 0, \ldots, 1, \ldots)$.
4. Clash? $\vec{p} \leftarrow (1-\beta)\vec{p} + \frac{\beta}{C-1}(1, 1, \ldots 0, \ldots)$.
5. Goto 2.

How Fast?

Unfinished after T steps with probability:

$$\left(1 - A\left((1-\beta)^2\left(\frac{\beta}{C-1}\right)^{2C+1}\right)^N\right)^{\frac{T}{N}}$$

Where A is the number of good colourings, N is number of nodes, C is number of colours.


office 001
office 002
office 003
office 004
office 005
office 006
Foyer
Seminar Room 009
office 008
office 007
office 017
office 016
office 010
Photocopying Room 015
office 014
office 011
office 012
store 013

Graph Variants

- Directed graphs.
- Hypergraphs.
- Better communication free colouring.
- Colouring in colouring dependent graphs.

Wrapping Up

- Modeling WiFi MAC.
- Markov chains.
- Why does approximation work?
- Channel allocation.
- Graph theory.
- New graph varients.