Geometry Tidbits

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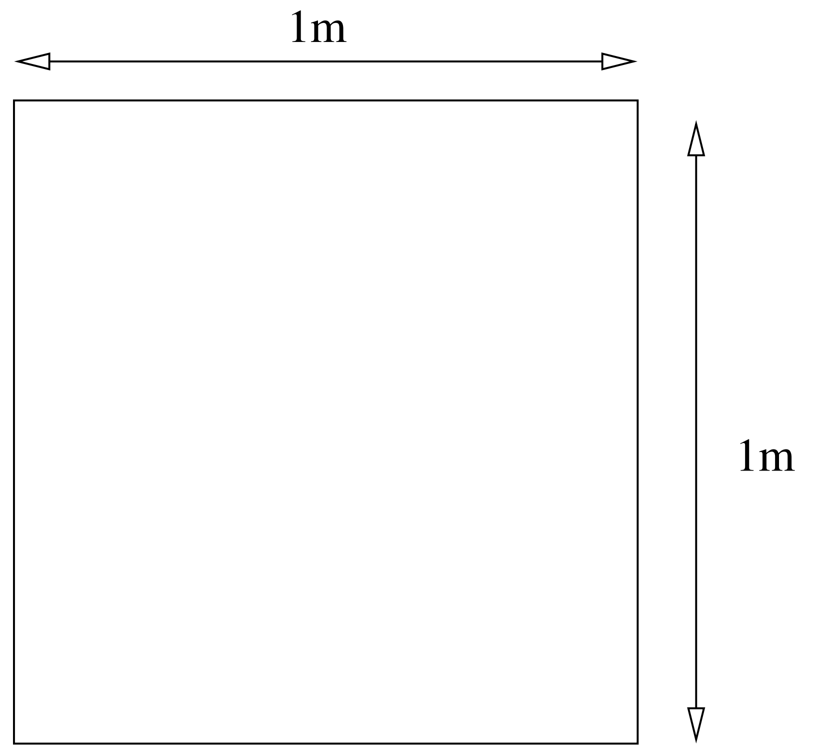
Tidbits

1. Ponds,
2. Rainbows,
3. Solids,
4. Cauchy-Schwartz,
5. n-Pyramids,
6. Inscribed Spheres,
7. Fractional Dimensions.

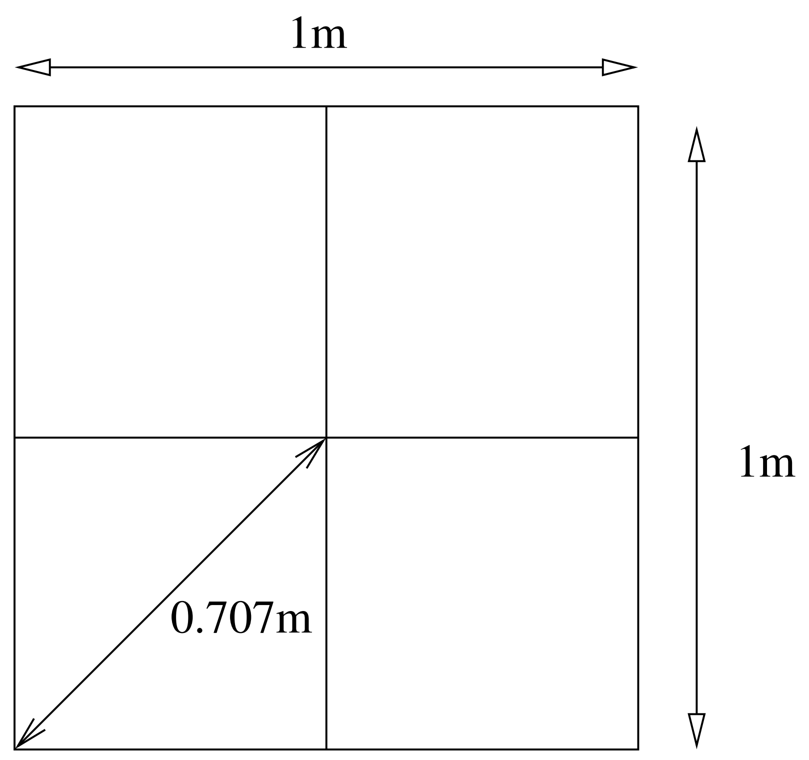
Ponds



5 Ducks are in this pond. Show that some two of them are within $1/\sqrt{2}m$ of one another.

Pigeon Hole Ducks

With n pigeon holes and $n+1$ pigeons, two pigeons live in same hole.

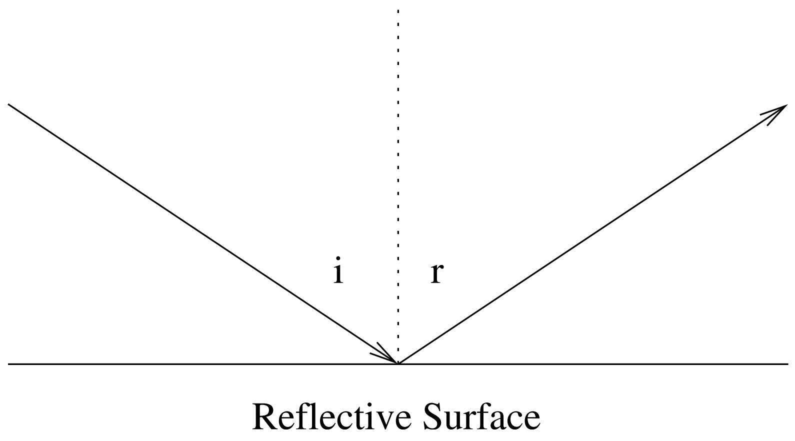


Rainbows

Try asking a physicist where rainbows come from.

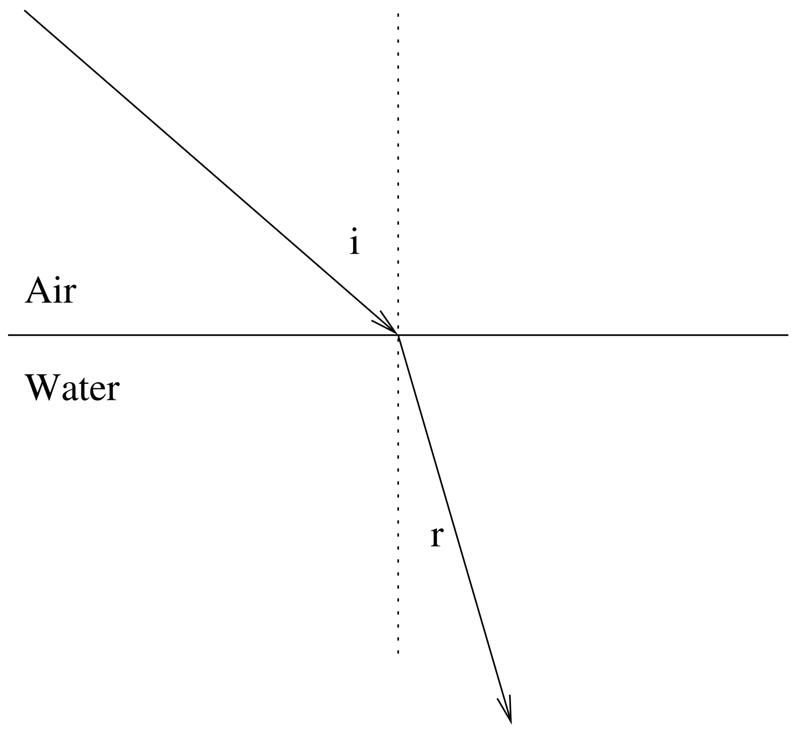
Rainbow Angle: $\approx 42^o$.

Reflection:



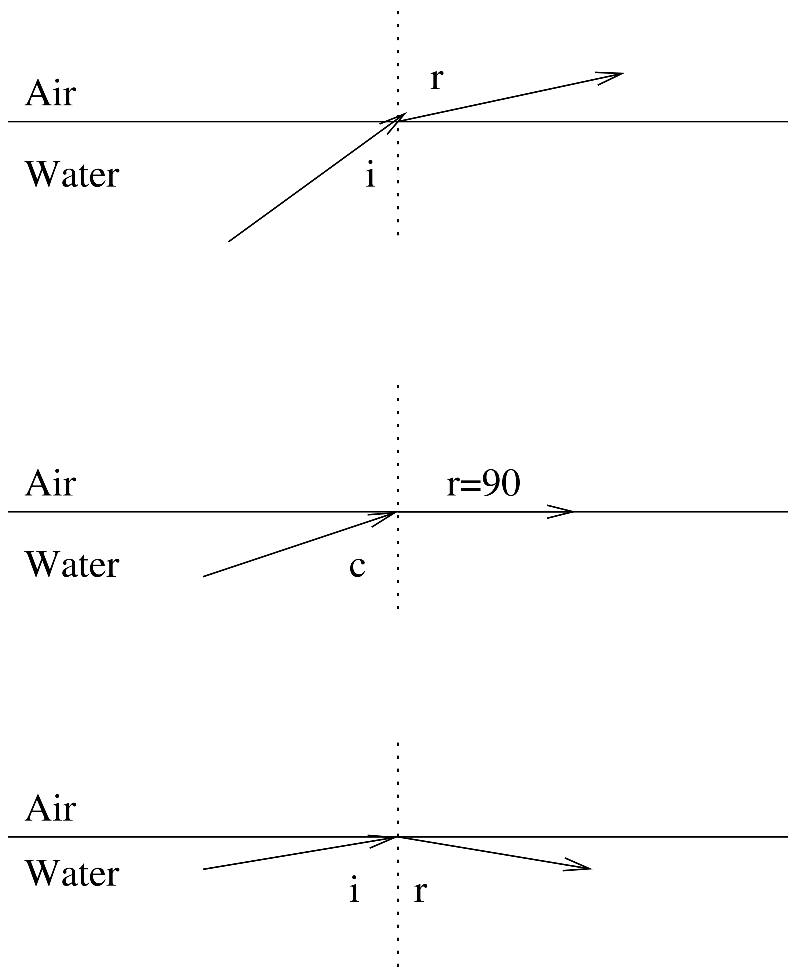
$$i = r \tag{1}$$

Refraction:



$$\frac{\sin(i)}{\sin(r)} = \frac{n_w}{n_a} \approx \frac{4}{3} \qquad (2)$$

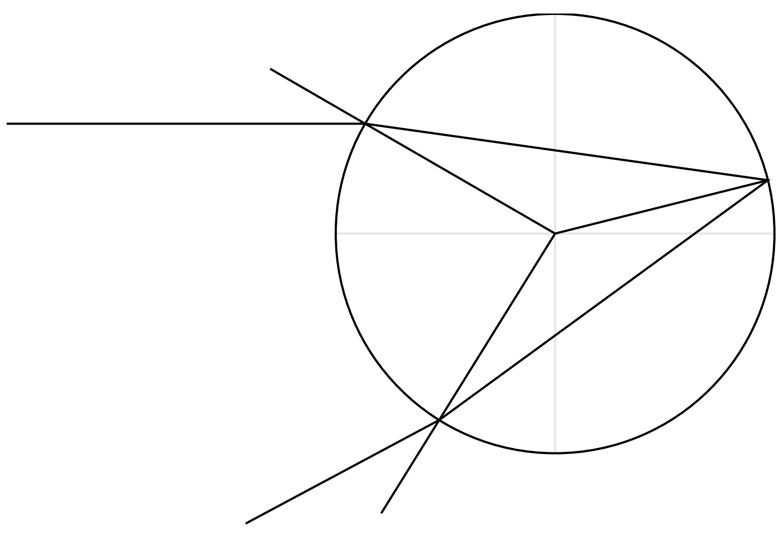
Total Internal Reflection:



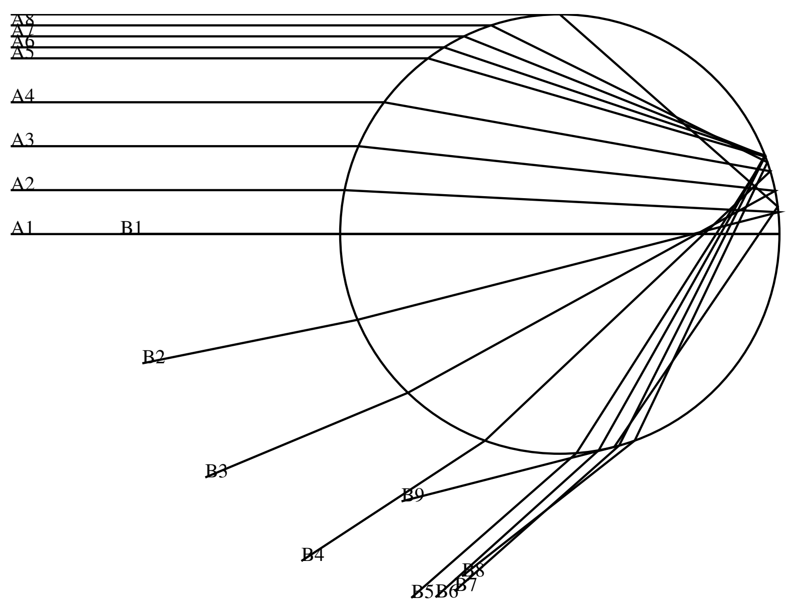
$$\frac{\sin(c)}{\sin(90)} = \frac{\sin(i)}{\sin(r)} = \frac{n_a}{n_w} \approx \frac{3}{4} \quad (3)$$

So $c \approx 48.6^o$.

Rainbows not related to TIR.



$$\delta = 4r - 2i \tag{4}$$



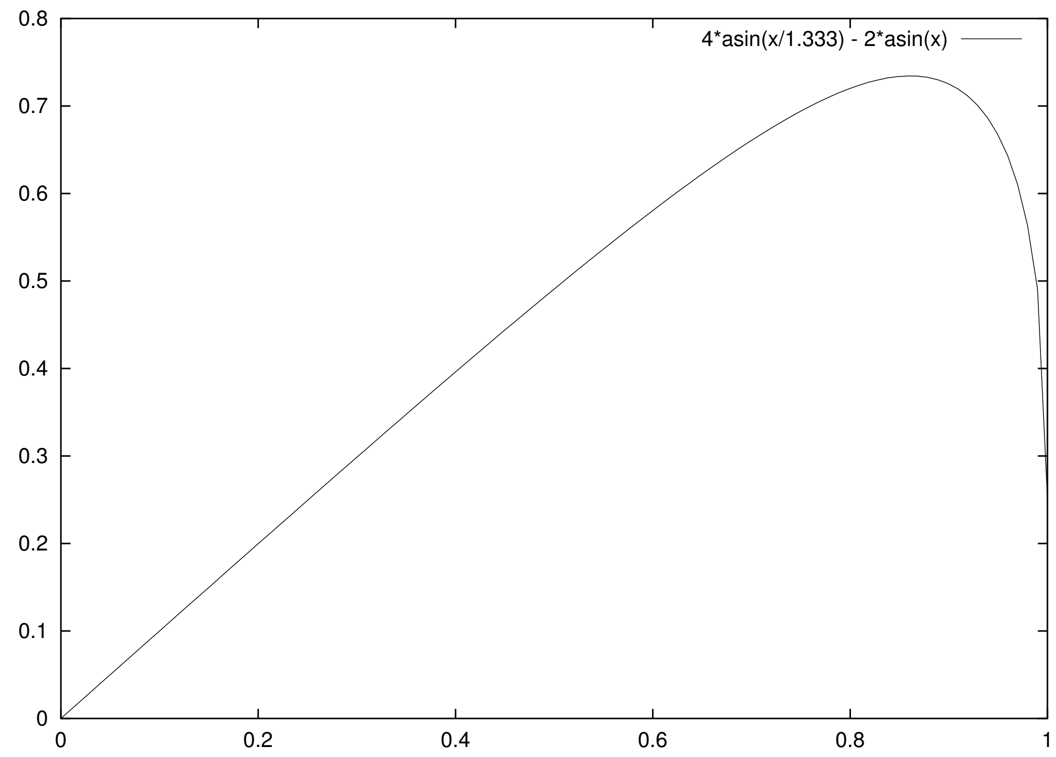
$$\delta = 4\sin^{-1}(l/n) - 2\sin^{-1}(l) \quad (5)$$

$$\frac{d\delta}{dl} = \frac{4}{\sqrt{n^2 - l^2}} - \frac{2}{\sqrt{1 - l^2}} \quad (6)$$

Turns at:

$$l = \sqrt{\frac{4 - n^2}{3}} \quad (7)$$

For water:



Or:

$$l = \sqrt{\frac{20}{27}} \approx 0.860\ldots \quad (8)$$

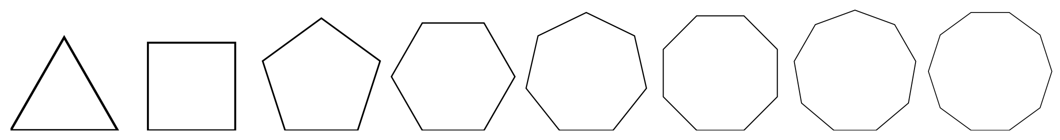
$$i = \sin^{-1}(l) \approx 59.4^o \quad (9)$$

$$r = \sin^{-1}(l/n) \approx 40.2^o \quad (10)$$

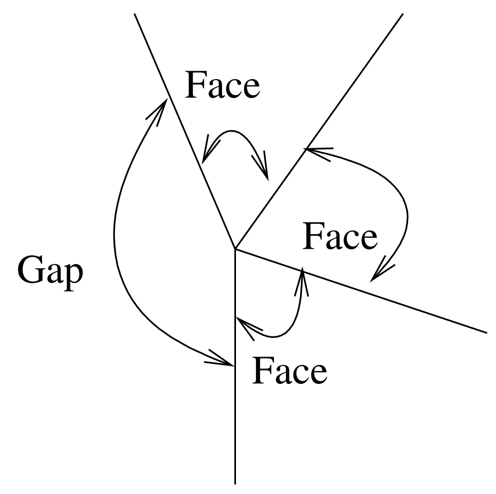
$$d = 4r - 2i \approx 42.3^o \quad (11)$$

Platonic Solids

Polyhedra in 2 dimensions:

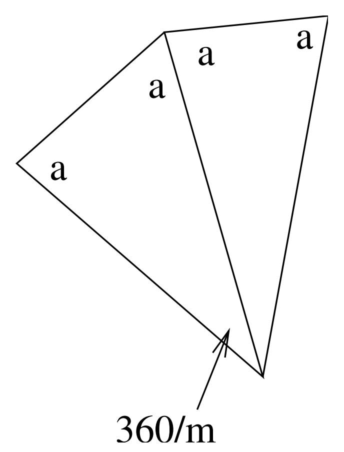


In three dimensions the situation is very different. There are only 5.



$n \geq 3$ faces at a corner:

$$n\theta < 360. \tag{12}$$



m edges per face:

$$\theta = 180 - \frac{360}{m} \tag{13}$$

Check cases for:

$$180 - \frac{360}{m} < \frac{360}{n} \tag{14}$$

$$\begin{aligned}
n = 3 \quad &\Rightarrow \quad 180 - \tfrac{360}{m} < 120 \\
&\Rightarrow \quad 60 < \tfrac{360}{m} \\
&\Rightarrow \quad m < 6 \\
&\qquad\qquad m = 3 \Rightarrow \text{tetrahedron} \\
&\qquad\qquad m = 4 \Rightarrow \text{cube} \\
&\qquad\qquad m = 5 \Rightarrow \text{dodecahedron} \\
n = 4 \quad &\Rightarrow \quad 180 - \tfrac{360}{m} < 90 \\
&\Rightarrow \quad 90 < \tfrac{360}{m} \\
&\Rightarrow \quad m < 4 \\
&\qquad\qquad m = 3 \Rightarrow \text{octahedron} \\
n = 5 \quad &\Rightarrow \quad 180 - \tfrac{360}{m} < 72 \\
&\Rightarrow \quad 108 < \tfrac{360}{m} \\
&\Rightarrow \quad m < 3.333\ldots \\
&\qquad\qquad m = 3 \Rightarrow \text{icosahedron}
\end{aligned}$$

No more ’cos $n \geq 6 \Rightarrow m < 3$.

Cauchy-Schwartz

For any $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$:

$$|a_1 b_1 + \ldots + a_n b_n| \le \sqrt{a_1^2 + \ldots + a_n^2}\sqrt{b_1^2 + \ldots + b_n^2} \quad (15)$$

Probably the most useful inequality in the world.

Neat derivation:

$$\vec{x} \cdot \vec{x} \ge 0$$

$$\left(\vec{a} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}\right) \cdot \left(\vec{a} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}\right) \ge 0$$

$$\vec{a} \cdot \vec{a} - 2\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{a} \cdot \vec{b} + \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)^2 \vec{b} \cdot \vec{b} \ge 0$$

$$\vec{a} \cdot \vec{a} \ge \frac{\left(\vec{a}\cdot\vec{b}\right)^2}{\vec{b}\cdot\vec{b}}$$

$$\sqrt{\vec{a} \cdot \vec{a}}\sqrt{\vec{b} \cdot \vec{b}} \ge \vec{a} \cdot \vec{b}$$

Link with geometry?

$$\left|\vec{a} \cdot \vec{b}\right| \leq |\vec{a}| \left|\vec{b}\right| \qquad (16)$$

Like to think of:

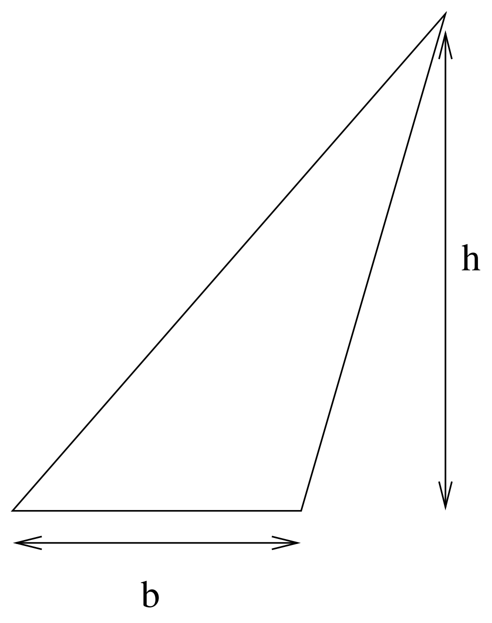
$$\vec{a} \cdot \vec{b} = |\vec{a}| \left|\vec{b}\right| \cos\theta \qquad (17)$$

So, Cauchy-Schwartz says:

$$\cos\theta \leq 1 \qquad (18)$$

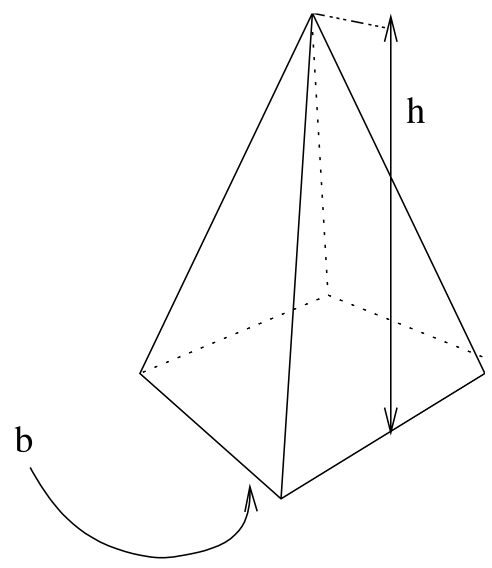
n-Pyramids

dim = 2

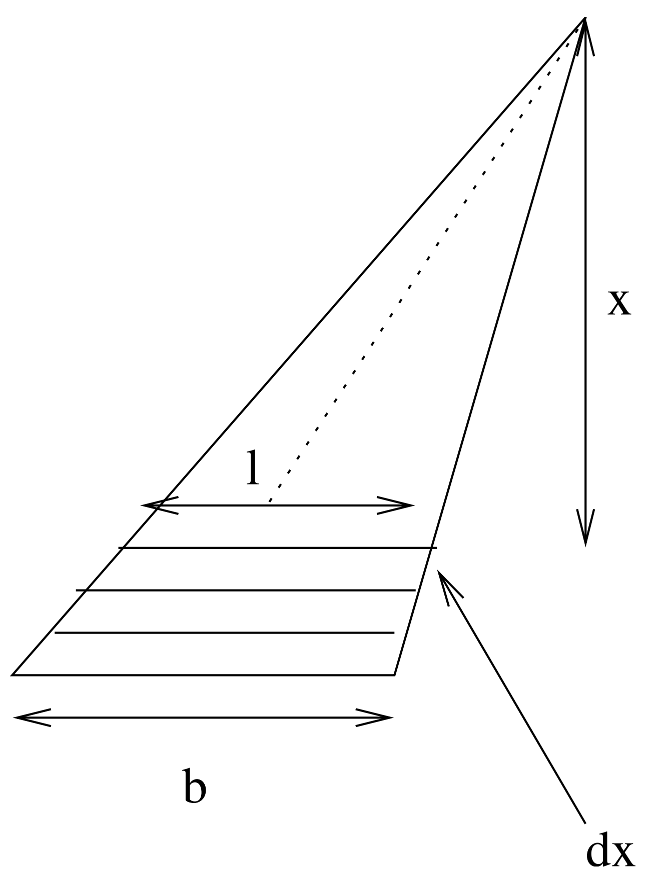


$$A = \frac{1}{2}bh$$

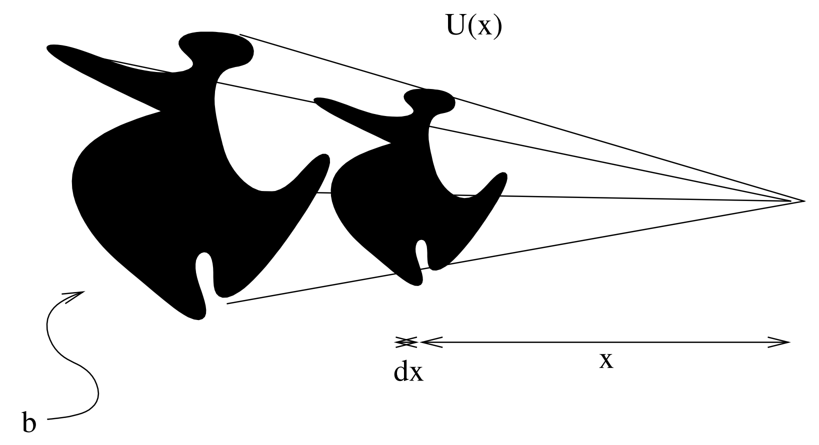
dim = 3



$$V = \frac{1}{3}bh$$



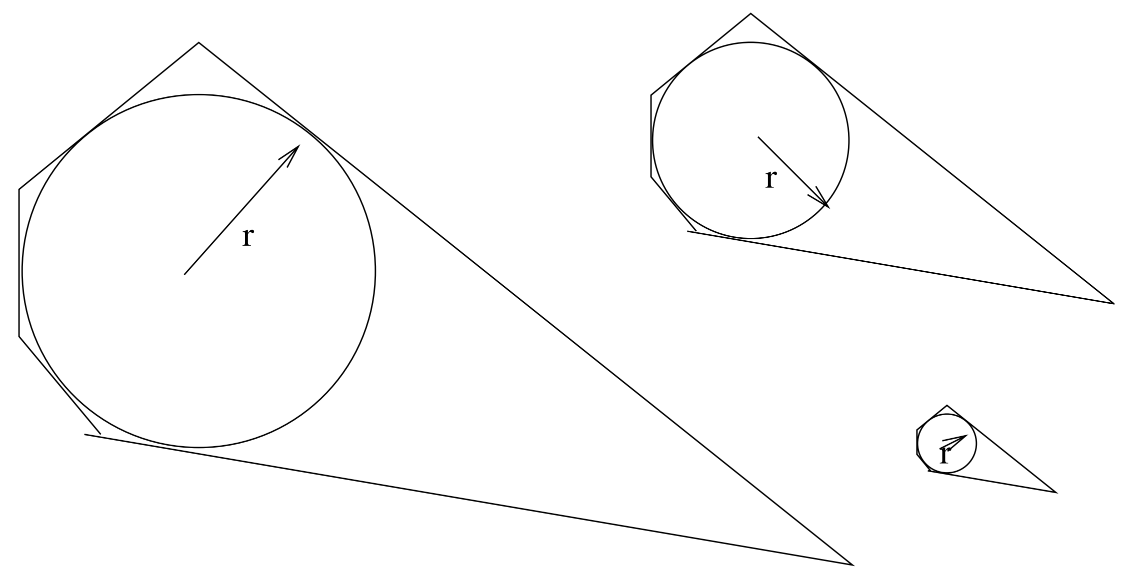
$$
\begin{aligned}
A &= \int_0^h l(x)\, dh \\
A &= \int_0^h b\frac{x}{h}\, dx \\
A &= \frac{b}{h}\int_0^h x\, dx \\
A &= \frac{b}{h}\frac{h^2}{2}
\end{aligned}
$$



$$
\begin{aligned}
V &= \int_0^h U(x)\,dh \\
A &= \int_0^h b\left(\frac{x}{h}\right)^{n-1}\,dx \\
A &= \frac{b}{h^{n-1}}\int_0^h x^{n-1}\,dx \\
A &= \frac{b}{h^{n-1}}\frac{h^n}{n} \\
A &= \frac{1}{n}bh
\end{aligned}
$$

Inscribed Sphere

Family of similar figures:



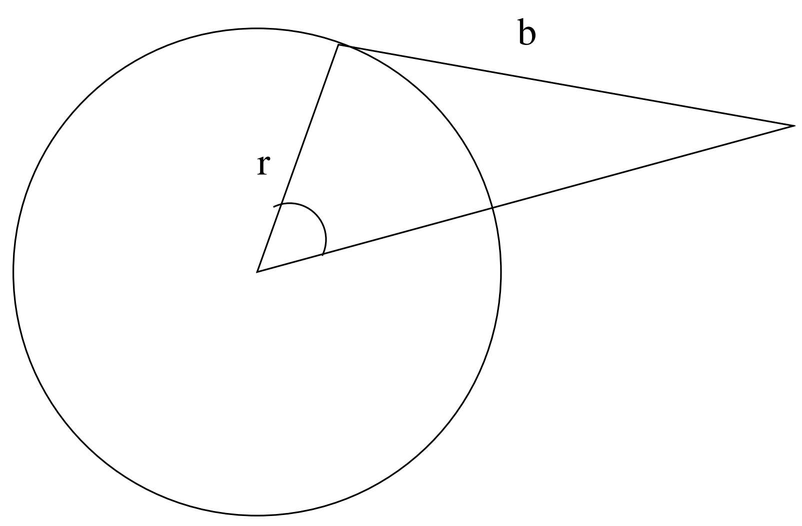
Write:

$$\text{Area} = A(r) \tag{19}$$

then:

$$\text{Perimiter} = \frac{dA}{dr} \tag{20}$$

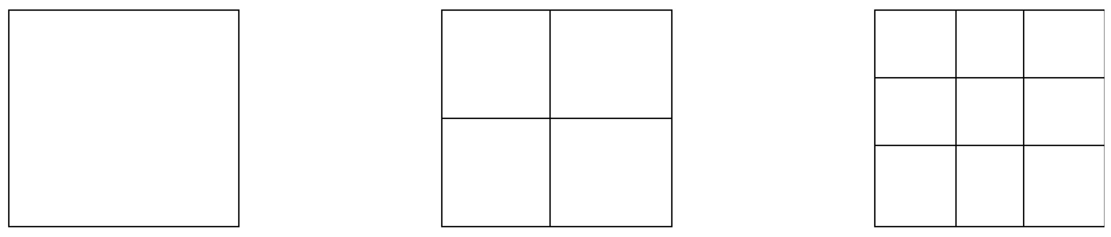
Figure made of triangular segments:



$$A = \sum A_i = \sum_i \tfrac{1}{2} b_i h_i = \sum_i \frac{1}{2} r r \tan \theta_i$$
$$P = \sum P_i = \sum_i b_i = \sum_i r \tan \theta_i$$

(Use pyramids in n dimensions.)

Fractional Dimensions

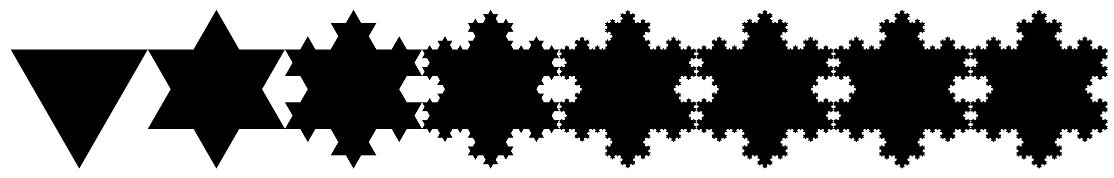


Scale by $1/n$ get $m = n^2$ pieces.

$$d = \log_n m \tag{21}$$

For example, use rule:

Split into $1/3$ and get 4 bits:



$$d = \log_3 4 \approx 1.2618\ldots \quad (22)$$

von Koch snowflake and Cantor set kind of opposits:

$$d = \log_3 2 \approx 0.6309\ldots \quad (23)$$

Postscript

```
%!PS-Adobe-2.0 EPSF-2.0
%%BoundingBox: 0 0 400 70
0 setgray
0 setlinewidth
/vk {
        1 3 div 1 3 div scale
        dup 0 gt
        {
                dup 1 sub vk
                60 rotate
                dup 1 sub vk
                -120 rotate
                dup 1 sub vk
                60 rotate
                dup 1 sub vk
        } {
                150 0 rlineto
        } ifelse
        3 3 scale
        pop
} def
/vktri {
        dup vk
        -120 rotate
        dup vk
        -120 rotate
        dup vk
        -120 rotate
        pop
} def
gsave newpath 0 50 moveto 0 vktri fill grestore
gsave newpath 100 50 moveto 2 vktri fill grestore
gsave newpath 200 50 moveto 4 vktri fill grestore
```