An Unusual Construction of the Fourier Transform

David Malone

December 6, 2000

Introduction

The usual construction of the Fourier transform involves working on $L^1(\mathbb{R})$ (eg. [1]) or the Schwartz Class $\mathcal{S}$ of rapidly decreasing C^∞ functions (eg. [2]). The Fourier transform is then extended onto $L^2(\mathbb{R})$ by taking limits as both $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\mathcal{S}$ are dense in $L^2(\mathbb{R})$.

I'd like to present an unusual construction of the Fourier transform in which we use its translation and dilation properties:

- if $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega}\hat{f}(\omega)$,
- if $g(x) = f(\lambda x)$ then $\hat{g}(\omega) = 1/|\lambda|\hat{f}(\omega/\lambda)$.

This construction is in the spirit of the "multiresolution analysis" structure [3] which is used to build discrete wavelet bases [4]. However, if you don't know anything about this structure the construction is still surprisingly straightforward.

The Definition

I'll be using both $\hat{f}$ and $\mathcal{F}f$ to denote the Fourier transform of f. For translation and dilation I'll write:

- $(\mathcal{T}_\alpha f)(x) := f(x + \alpha)$ for any $\alpha \in \mathbb{R}$,
- $(\mathcal{D}_\lambda f)(x) := f(\lambda x)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$,
- $(\mathcal{R}_\alpha f)(x) := e^{i\alpha x}f(x)$ for any $\alpha \in \mathbb{R}$.

This allows me to write the translation and dilation properties as $\mathcal{F}\mathcal{T}_\alpha f = \mathcal{R}_\alpha \mathcal{F}f$ and $\mathcal{F}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{F}f$.

Now to define $\mathcal{F}$ we do the following:

1. for the characteristic function of $[0,1)$ we define $\mathcal{F}(\chi_{[0,1)}) = \frac{1-e^{-i\omega}}{i\omega}$,
2. we extend $\mathcal{F}$ to any $\chi_{[n,n+1)}$ by using the translation rule $\mathcal{F}\mathcal{T}_n f = \mathcal{R}_n \mathcal{F} f$ for all $n \in \mathbb{Z}$,
3. we further extend $\mathcal{F}$ to functions of the form $\chi_{[\lambda^{-1}n,\lambda^{-1}(n+1))}$ by using the dilation rule $\mathcal{F}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{F} f$ for all $\lambda \in 2^{\mathbb{Z}}$,
4. finally we extend $\mathcal{F}$ to the linear span of these by assuming $\mathcal{F}$ is linear.

If this process works we have defined $\mathcal{F}$ on D the set of dyadic step functions. These are just the simple functions whose jumps occur at $n/2^m$ where $n, m \in \mathbb{Z}$. It is easy to construct a well defined function which has these properties. In fact the definition spells out a formula:

$$f(x) = \sum_{r=-R}^{R} a_r \chi_{[0,1)}(2^j x - r) \quad \Rightarrow \quad \mathcal{F}f(\omega) = \frac{1 - e^{-i\omega/2^j}}{i\omega} \sum_{r=-R}^{R} a_r e^{-ir\omega/2^j}.$$

Our aim was to produce $\mathcal{F}$ on $L^2(\mathbb{R})$. Given that the set of simple functions is dense in $L^2(\mathbb{R})$ it is clear that D is also dense in $L^2(\mathbb{R})$. So, if we can show this function $\mathcal{F}$ we have defined is continuous in the $L^2(\mathbb{R})$ norm then we can extend $\mathcal{F}$ to all of $L^2(\mathbb{R})$.

This turns out to be surprisingly straight forward. Taking

$$f(x) = \sum_{r=-R}^{R} a_r \chi_{[0,1)}(2^j x - r)$$

we see that $\|f\|_2^2 = \sum_{r=-R}^{R} |a_r|^2/2^j$. Now we have to find $\|\mathcal{F}f\|_2^2$. Using our formula above and the definition of the norm:

$$\begin{aligned} &\|\mathcal{F}(f)\|_2^2 \\ = &\int_{-\infty}^{\infty} \left| \frac{1 - e^{-i\frac{\omega}{2^j}}}{i\omega} \right|^2 \left(\sum_{k=-N}^{N} a_k e^{-ik\frac{\omega}{2^j}} \right) \left(\sum_{l=-N}^{N} \overline{a_l} e^{il\frac{\omega}{2^j}} \right) d\omega \\ = &\int_{-\infty}^{\infty} \frac{2(1 - \cos\frac{\omega}{2^j})}{\omega^2} \left[\left(\sum_{k=-N}^{N} |a_k|^2 \right) + \left(\sum_{k \neq l} a_k \overline{a_l} e^{-i(k-l)\frac{\omega}{2^j}} \right) \right] d\omega. \end{aligned}$$

So we need to evaluate:

$$\int_{-\infty}^{\infty} \frac{2(1 - \cos\omega)}{\omega^2} e^{ir\omega}\, d\omega,$$

for $r \in \mathbb{Z}$. This is an easy piece of contour integration, giving 2π if $r = 0$ and zero otherwise. Filling this in we see:

$$\|\mathcal{F}f\|_2^2 = \sum_{r=-R}^{R} |a_r|^2/2^j 2\pi = 2\pi\|f\|_2^2.$$

So, not only is $\mathcal{F}$ continuous but it just scales the norm. This means that we may extend $\mathcal{F}$ to a continuous map from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ which preserves the inner product:

$$(f, g) = 2\pi(\mathcal{F}f, \mathcal{F}g).$$

What now?

Note that we could show that $\mathcal{F}$ as defined on D also extends to a continuous map $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ by examining the $L^1(\mathbb{R})$ norm of f and the $L^\infty(\mathbb{R})$ norm of $\mathcal{F}f$. This might motivate us to try to get the usual integral formula for the Fourier transform back again.

This can be done for $f \in D$ by first considering f as a function with steps of width 2^{-j}, and then splitting each step in half to get the same function written in terms of steps of width $2^{-(j+1)}$. This turns our formula for $\mathcal{F}$ into a Riemann sum for the integral:

$$(\mathcal{F}f)(\omega) = \int f(x)e^{-i\omega x}\,dx \qquad f \in D.$$

This can naturally be extended to suitable sets larger than D.

By looking at the dilation and translation relations carefully (or by using the integral formula) we get extended translation and dilation, this time for all $f \in L^2(\mathbb{R})$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$:

1. $\mathcal{F}\mathcal{T}_\alpha f = \mathcal{R}_\alpha \mathcal{F}f$,
2. $\mathcal{F}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{F}f$,
3. $\mathcal{F}\mathcal{R}_\alpha f = \mathcal{T}_{-\alpha}\mathcal{F}f$.

This provides us with a neat way to show that $\mathcal{F}$ is invertible on $L^2(\mathbb{R})$. Suppose we defined $\mathcal{G}$ with the translation and dilation properties we expect of $\mathcal{F}^{-1}$. Then by proceeding as we did for $\mathcal{F}$ we arrive at an integral formula and the following properties for $\mathcal{G}$:

1. $\mathcal{G}\mathcal{T}_\alpha f = \mathcal{R}_{-\alpha}\mathcal{G}f$,
2. $\mathcal{G}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{G}f$,
3. $\mathcal{G}\mathcal{R}_\alpha f = \mathcal{T}_\alpha \mathcal{G}f$.

We examine $\mathcal{I} = \mathcal{F}\mathcal{G}$ and how it interacts with $\mathcal{T}_n$, $\mathcal{D}_\lambda$ and $\chi_{[0,1)}$. Using the algebraic properties of $\mathcal{F}$ and $\mathcal{G}$:

$$\mathcal{I}\mathcal{T}_n f = \mathcal{F}\mathcal{G}\mathcal{T}_n f = \mathcal{F}\mathcal{R}_{-n}\mathcal{G}f = \mathcal{T}_n\mathcal{F}\mathcal{G}f = \mathcal{T}_n\mathcal{I}f$$

$$\mathcal{I}\mathcal{D}_\lambda f = \mathcal{F}\mathcal{G}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{F}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{G}f = \frac{|\lambda|}{|\lambda|}\mathcal{D}_\lambda\mathcal{F}\mathcal{G}f = \mathcal{D}_\lambda\mathcal{I}f$$

Thus $\mathcal{I}$ commutes with $\mathcal{T}_n$ and $\mathcal{D}_\lambda$, so if we can determine the image of $\chi_{[0,1)}$ we can determine the image of D. Using the integral formula and a little contour integration we see:

$$(\mathcal{I}\chi_{[0,1)})(x) = (\mathcal{F}\mathcal{G}\chi_{[0,1)})(x) = \mathcal{F}(\frac{e^{i\omega}-1}{i\omega}) = \int \frac{e^{i\omega}-1}{i\omega} e^{-i\omega x}\, d\omega = 2\pi\chi_{[0,1)}(x)$$

for almost every x. So $\mathcal{I}$ acts on D by multiplying by 2π. Using the fact that $\mathcal{I}$ is continuous we see that $\mathcal{I}$ acts on all of $L^2(\mathbb{R})$ in this way, and so $(2\pi)^{-1}\mathcal{G}$ is a right inverse for $\mathcal{F}$. Naturally a similar argument shows that it is also a left inverse.

To finish up

This is a curious construction of the Fourier transform. It is even quite easy to extend it to $L^2(\mathbb{R}^n)$. One interesting point I didn't touch on is that we may change the first rule we defined $\mathcal{F}$ with from:

- for the characteristic function of $[0,1)$ we define $\mathcal{F}(\chi_{[0,1)}) = \frac{1-e^{-i\omega}}{i\omega}$,

to the seemingly weaker:

- $\mathcal{F}(\chi_{[0,1)})$ is continuous at zero and has value 1 at zero.

This is because $\chi_{[0,1)}$ satisfies the dilation equation: $\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1)$, but that is another story [5].

References

[1] Elias M. Stein and Guido Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, (1971).

[2] H. Dym and H. P. KcKean, *Fourier Series and Integrals*, Academic Press, (1972).

[3] Yves Meyer, *Wavelets and operators*, Cambridge University Press, (1992).

[4] Ingrid Daubechies, *Ten Lectures on Wavelets*, SIAM, (1992).

[5] David Malone, *Fourier Analysis, Multiresolution Analysis and Dilation Equations*, Masters Thesis, University of Dublin, (1997).