Irish MO Training: Functional Equations

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Equations

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$$x^2 - 2x - 3 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm 4}{2} = -1 \text{ or } 3$$

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$$x = 2, y = 1$$

Functions

A function is a rule that transforms one object into another in a consistent way.

$$
\begin{aligned}
E(A) &= 1, E(B) = 2, \ldots E(Z) = 26 \\
R(t) &= R_0 2^{-\frac{t}{H}} \\
P(x) &= 3x^x + 2x^2 + x^1 + 0 \qquad \text{polynomial} \\
R(x) &= \frac{x+1}{x^2+1} \qquad \text{rational} \\
A(b,h) &= \frac{hb}{2} \\
\phi(n) &= \text{number of positive integers} \leq n \text{ that are relatively prime to } n
\end{aligned}
$$

Might write a_n instead of $a(n)$ for a sequence of integers.

Functional Equations

Functional Equations are equations you solve to find a function rather than a number.

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$$f(x) = \frac{7x - 4}{3}$$

Functional equations are all over mathematics. Equations involving differentiation are common in Physics:

$$\frac{d^2 f}{dx^2}(t) + f(t) = 0$$

(Here the solution is $f(t) = a\sin(t) + b\cos(t)$.)

$$a_{n+1} = 3a_n - a_{n-1}$$

Sometimes there are standard ways to solve a functional equation. Other times you have to guess.

Difference Equations

A difference equation expresses the next term (a_{n+1}) in a sequence as a sum of previous terms:

$$a_{n+1} = Aa_n + Ba_{n-1} + Ca_{n-2} \ldots$$

You need to know the first few terms to get going.
For example, suppose $F_n = F_{n-1} + F_{n-2}$ and $F_1 = 1$ and $F_2 = 1$.
Then the sequence is $1, 1, 2, 3, 5, 8, \ldots$.

There is a standard way to solve difference equations, as long as $A, B, C, \dots$ are fixed numbers that don't depend on n or the a_ns. Steps:

1. Replace a_n with x^n.
2. You get an equation — find the roots $x, y, \dots$.
3. The answer will be

   $$Dx^n + Ey^n + \dots$$

   but you have to find $D, E, \dots$ to match the first few terms you were given.

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Example

Let's solve

$$F_{n+1} = F_n + F_{n-1}$$

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Divide through by x^{n-1} to get $x^2 = x + 1$. So

$$x^2 - x - 1 = 0$$

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Divide through by x^{n-1} to get $x^2 = x + 1$. So

$$x^2 - x - 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{5}}{2}$$

So our solution looks like:

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

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$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

We need to find A and B using the information $F_1 = 1$ and $F_2 = 1$.

$$F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$F_2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^2 = \left(\frac{3 \pm \sqrt{5}}{2}\right)$$

$$F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$F_2 = A\left(\frac{3+\sqrt{5}}{2}\right)^2 + B\left(\frac{3-\sqrt{5}}{2}\right)^2 = 1$$

Gathering up terms

$$\frac{A+B}{2} + \frac{A-B}{2}\sqrt{5} = 1$$

$$\frac{A+B}{2}3 + \frac{A-B}{2}\sqrt{5} = 1$$

So $A + B = 0$, then $B = -A$ and $(A + A)/2\sqrt{5} = 1$, or $A = 1/\sqrt{5}$.

So our solution is

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

The steps were:

1. Replace a_n with x^n.
2. You get $x^2 - x - 1 = 0$, find the two roots x, y.
3. The answer will be

   $$Ax^n + By^n$$

   but you have to find A, B so first two terms are 1.

Puzzle

How many digits does F_{2000} have?
Hint: $\log_{10}\left((1+\sqrt{5})/2\right) \approx 0.20898764$.

We'll come back to this to give you a chance to think.

Another example

I said things were a little more complex if you get a double root. Let's have an example.

Find a formula for $a(n)$ if $a(1) = 5$, $a(2) = 13$ and $a(3) = 13$ and $a(n) = a(n-1) + a(n-2) - a(n-3)$.

First replace $a(n)$ with x^n:

$$x^n = x^{n-1} + x^{n-2} - x^{n-3}$$

$$x^3 - x^2 - x + 1 = 0$$

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$$(x^2 - 1)(x - 1) = 0$$

$$x^3 - x^2 - x + 1 = 0$$

$$x^2(x - 1) - (x - 1) = 0$$

$$(x^2 - 1)(x - 1) = 0$$

$$(x + 1)(x - 1)(x - 1) = 0$$

$$x^3 - x^2 - x + 1 = 0$$

$$x^2(x-1) - (x-1) = 0$$

$$(x^2 - 1)(x-1) = 0$$

$$(x+1)(x-1)(x-1) = 0$$

So 1 is a double root and -1 is a single root.

$$a(n) = (-1)^n A + 1^n \ B$$

Because $a(n) = a(n-1) + a(n-2) - a(n-3)$, we need three values to get started and need three unknowns to match.

$$a(n) = (-1)^n A + 1^n(B + Cn)$$

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$$a(1) = -A + B + 1C = 5$$

$$a(2) = +A + B + 2C = 13$$

$$a(3) = -A + B + 3C = 13$$

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Subtract first and last get $2C = 8,\ C = 4$.

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$-A + B = 1$ and $A + B = 5$.
So $A = 2$ and $B = 3$.

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Subtract first and last get $2C = 8$, $C = 4$.
$-A + B = 1$ and $A + B = 5$.
So $A = 2$ and $B = 3$.

$$a(n) = (-1)^n 2 + 3 + n4.$$

Puzzle

How many digits does F_{2000} have?
Hint: $\log_{10}\left((1+\sqrt{5})/2\right) \approx 0.20898764$.

$$F_{2000} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{2000} - \left(\frac{1-\sqrt{5}}{2}\right)^{2000}\right)$$

Roughly

$$F_{2000} = \frac{1}{\sqrt{5}}\left((1.618)^{2000} - (-0.618)^{2000}\right)$$

The second term will be tiny! So it is mainly

$$\log_{10} F_{2000} = -0.5\log_{10} 5 + 2000\log_{10}\left((1+\sqrt{5})/2\right)$$

$$\log_{10} F_{2000} = -0.5 \log_1 05 + 2000 * 0.20898764 = 417.97528 - \text{small}$$

Finally, 10–99 have $\log_{10}$ between 1 and 2, 100–999 between 2 and 3. So F_{2000} has 418 digits.

Other Techniques

All sorts of techniques can work.
Find all functions on the positive integers so that:

$$f(2n) = 2f(n)$$

and

$$f(2n+1) = 2f(n) + f(1)$$

Let's try a few small values.

Could now use Induction

Want to show $f(n) = nc$.
First check $n = 1$:

$$f(1) = c = 1c.$$

Now, assume true for $n = 1, 2, \dots k - 1$. Want to show true for $k \geq 2$.
If k is even, then $k = 2l$ and l will be bigger than 1 but less than k. Then

$$f(k) = f(2l)$$

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If k is odd, then $k = 2l + 1$ and l will be bigger than 1 but less than k. Then

$$f(k) = f(2l + 1) = 2f(l) + f(1) = 2lc + c = (2l + 1)c = kc.$$

Sometimes There's No Technique

A function f mapping the set of positive integers into itself satisfies

- $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
- $f(p+q) = f(p) + f(q)$ for all prime numbers p, q.

Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

A good way to start any question is to try a few small values.