2 and 3 Refinable Functions

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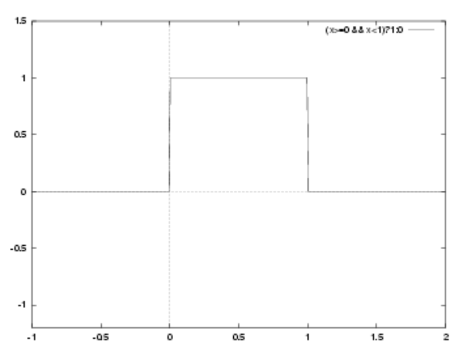
Refinable Functions

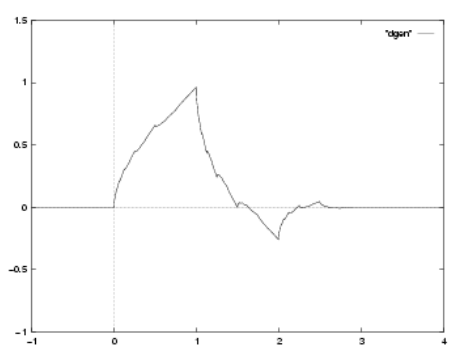
A 2-refinable function satisfies:

$$f(x) = \sum_k c_k f(2x - k).$$

The equation is a dilation equation.



Harr function



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People are usually interested in compactly supported L^1 solutions for wavelets with $\sum c_k = 2$. In this case there is at most one solution.

L^2 solutions are not unique. They are in correspondence with $L^2(\pm[1,2))$ or $L^\infty(\pm[1,2))$. However, adding more scales gave uniqueness in operator results.

What functions are 2 and 3 refinable? ie.

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k).$$

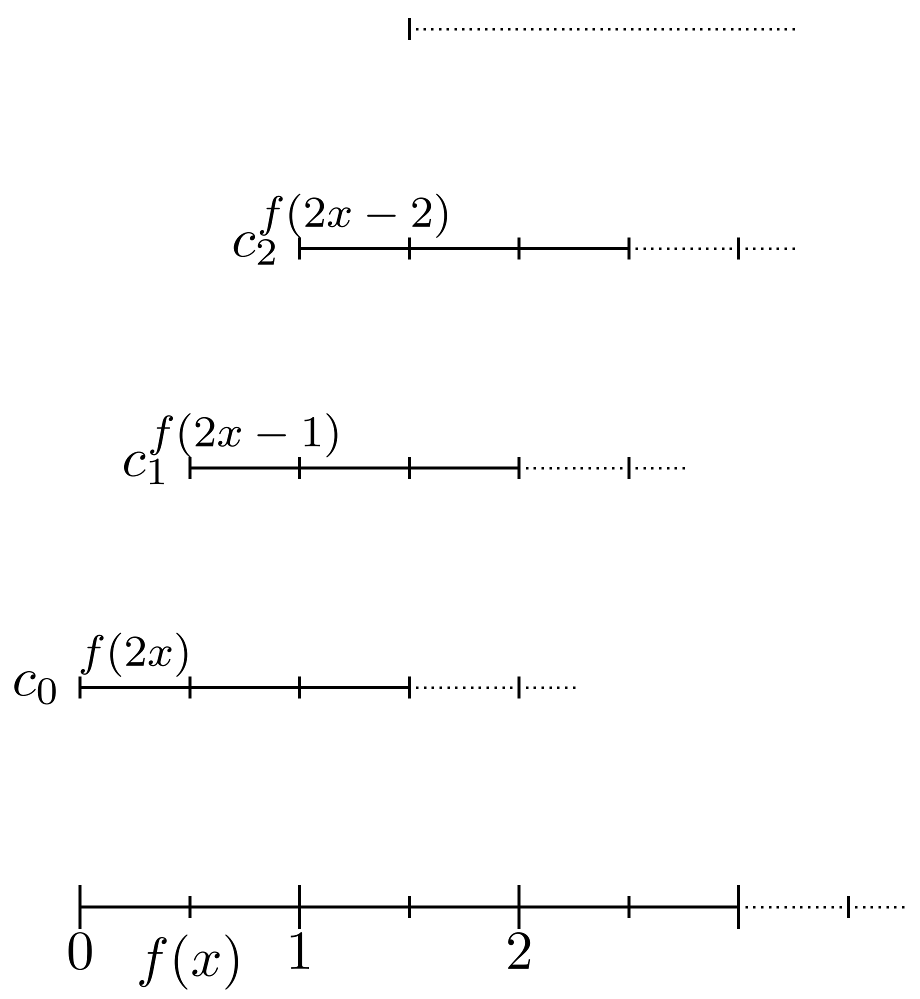
Assumptions:

- Only finitely many non-zero c_k and d_k.
- Functions compactly supported.

Left Hand End

Lemma 1 *Suppose* $g(x) = \sum d_k g(2x - k)$*, and only finitely many of the* d_k *are non-zero. Then we can find* l *so that when we translate* g *by* l *to get* f *we find:* $f(x) = \sum c_k f(2x - k)$*,* $c_0 \neq 0$*,* $c_k = 0$ *when* $k < 0$ *and* $c_k = d_{k-l}$*.*

Lemma 2 *If* f *is compactly supported and satisfies a dilation equation* $f(x) = \sum c_k f(2x - k)$*, where* $c_0 \neq 0$ *and* $c_k = 0$ *when* $k < 0$*, then* f *is zero almost everywhere in* $(-\infty, 0)$*.*



$$
\begin{aligned}
f(x) &= \sum_{k=0}^{n} c_k f(2x-k) \\
&\qquad x \in \left[\frac{n}{2}, \frac{n+1}{2}\right) \\
f(x) &= \frac{f\left(\frac{x}{2}\right) - \sum_{k=1}^{n} c_k f(x-k)}{c_0} \\
&\qquad x \in [n, n+1).
\end{aligned}
$$

If you do this for two scales, everything lines up and we get:

$$f(x) = c_0 f(2x) = d_0 f(3x),$$

on $[0, 1/2) \cap [0, 1/3)$.

Theorem 3 *Suppose f is 2 and 3 refinable, say:*

$$f(x) = \sum_k c_k f(2x-k) = \sum_k d_k f(3x-k),$$

and $c_0 \neq 0$ and $c_k = 0$ when $k < 0$. Suppose further that f is integrable on some interval $[0, \epsilon]$, then $f(x) = \gamma x^\beta$ on $[0, 1)$ where $\beta = -\log_2 c_0 = -\log_3 d_0$.

In L^p this will be an almost everywhere relationship. So we integrate:

$$F(x) = \int_0^x f(t)\,dt.$$

We can show this satisfies:

$$F(2^n 3^m \alpha) = \left(\frac{2}{c_0}\right)^n \left(\frac{3}{d_0}\right)^m F(\alpha)$$

Continuity forces $\log_2 c_0 = \log_3 d_0 = -\beta$. So:

$$F(2^n 3^m \alpha) = (2^n 3^m)^{\beta+1} F(\alpha)$$

One continuous function does this:

$$x^{\beta+1} \frac{F(\alpha)}{\alpha^{\beta+1}}$$

Giving f on $[0, 1/3)$:

$$f(x) = \frac{d}{dx}F(x) = (\beta + 1)x^\beta \frac{F(\alpha)}{\alpha^{\beta+1}}.$$

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From here it is easy so show f must have the form:

$$f(x) = \sum_{l=0}^{n} a_l (x - l)^\beta,$$

on $[n, n+1)$. For this to be compactly supported $\beta \in \mathbb{N}$. This means that f must be a B-spline.

The fact that f behaves like x^β on $[0, 1)$ is interesting, and actually holds in a much looser sense if you have function which just solves one dilation equation.

There is a measure of smoothness called the Hölder exponent, where $f \in C^{n+s}$ if f is n times differenciable and:

$$|f(x + h) - f(x)| < k|h|^s.$$

Now, $x^\beta \in C^{n+s}$ for $n + s < \beta$, and it can be shown that for $f \in C^{n+s}$ then $n + s < \beta$.

I have examples which can produce f which are almost this smooth, and in some simple cases this gives the correct smoothness.

	Coefficents		TLoW			SRCfSS		
N	$\lvert c_0 \rvert$	$-\log_2 \lvert c_0 \rvert$	p226	p232	p239	r_{20}	r_∞	UB
2	0.683	0.550	0.339	0.500	0.550	0.550	0.550	0.550
3	0.470	1.0878	0.636	0.915	1.0878	1.0831	1.0878	1.0878
4	0.325	1.6179	0.913	1.275	1.6179	1.6066	1.6179	1.6179
5	0.2264	2.1429	1.177	1.596		1.9424		1.9689
6	0.1577	2.6644	1.432	1.888		2.1637		2.1891
7	0.1109	3.1831	1.682	2.158		2.4348		2.4604