A Multi-Resolution Approach to Fourier Analysis

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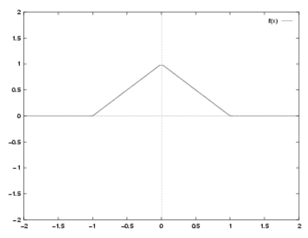
TCD

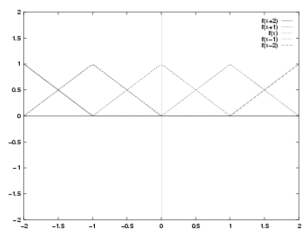
Multi-Resolution Analysis

To approximate something we:

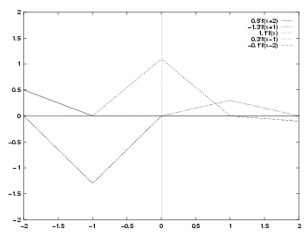
1. Take some "basis" function g.



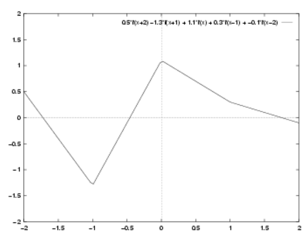
2. Move it to our nodes.



3. Multiply by some coefficients.



4. Sum the results.



This is our first step in a multi-resolution analysis.

1. Our first resolution of approximation is:

$$V_0 = \operatorname{span}\{g(x-n) : n \in \mathbb{Z}\}$$

2. The $g(x-n)$ should be orthogonal.
3. Now build many levels:

$$f(x) \in V_k \iff f(2x) \in V_{k+1}$$

4. They should be increasing:

$$V_k \subset V_{k+1}$$

5. The union should be dense:

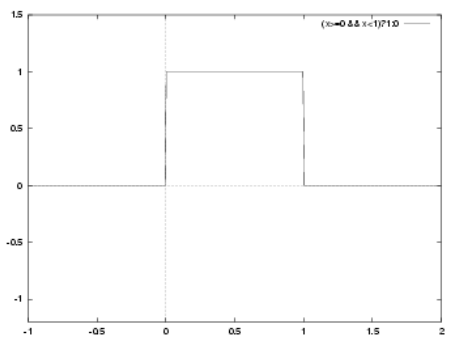
$$\overline{\bigcup_{j=-\infty}^{+\infty}} V_j = L^2(\mathbb{R})$$

6. The intersection should be almost empty:

$$\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$

3 Examples

1. $g(x) = \chi_{[0,1)}$.



2. $\widehat{V_j}$ is functions supported on $[-2^j\pi, 2^j\pi]$.
   Using:

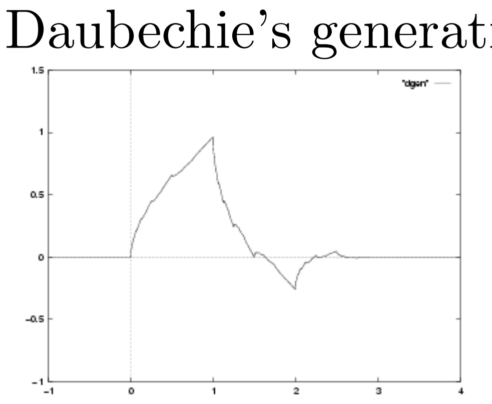
$$\mathcal{F}\left(f(2x)\right)(\omega) = \frac{1}{2}\mathcal{F}\left(f(x)\right)\left(\frac{\omega}{2}\right)$$

   We can check:

$$f(x) \in V_k \iff f(2x) \in V_{k+1}$$

   Also $g(x) = \frac{\sin \pi x}{\pi x}$ works.

3. Daubechie's generating Function.



Why MRA ?

- Natural way to increase resolution.
- Produces wavelets.

$$V_{j+1} = V_j \oplus W_j$$

- Many properties relate to g chosen.
- Dilation Equations.

$$g \in V_0 \subset V_1$$

but V_1 is the span of $g(2x - k)$, so:

$$g(x) = \sum_k c_k g(2x - k)$$

The c_k contain important information about g.

How to define: Determinants

Formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Properties

$$\det : \mathbb{R}^{n \times n} \to \mathbb{R}$$

- $\det(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) = 1$ where $\{\vec{e}_j\}_{j=1..n}$ is the usual basis for $\mathbb{R}^n$.
- $\det(\ldots, \vec{a}_i, \ldots, \vec{a}_j, \ldots) = \det(\ldots, \vec{a}_i + \vec{a}_j, \ldots, \vec{a}_j, \ldots)$
- $\det(\lambda_1 \vec{a}_1, \lambda_2 \vec{a}_2, \ldots, \lambda_n \vec{a}_n) = (\lambda_1 \lambda_2 \ldots \lambda_n) \det(\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n)$

How to define: Fourier Transform

Formula

$$\mathcal{F}(f)(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} f(x)\,dx$$

Properties

$$g(x) = \chi_{[0,1)}(x)$$

1. $\mathcal{F}\left(g\right)\left(\omega\right) = \frac{1-e^{-i\omega}}{i\omega}$
2. $\mathcal{F}\left(\mathcal{T}_n f\right)\left(\omega\right) = e^{i\omega n}\mathcal{F}\left(f\right)\left(\omega\right)$ for $n \in \mathbb{Z}$.
3. $\mathcal{F}$ is linear.
4. $\mathcal{F}\left(\mathcal{D}_\lambda f\right)\left(\omega\right) = \frac{1}{|\lambda|}\mathcal{F}\left(f\right)\left(\frac{\omega}{\lambda}\right)$ where $\lambda = 2^n, n \in \mathbb{Z}$.

$$\mathcal{T}_n f(x) = f(x+n)$$

$$\mathcal{D}_\lambda f(x) = f(\lambda x)$$

The Plan

We'll call $\bigcup_{j=-\infty}^{+\infty} V_j = D$.

Check $\mathcal{F}$ is well defined. Some f in D have more than one representation

Extend $\mathcal{F}$ to all of $L^2(\mathbb{R})$. To take limits we must show $\mathcal{F}$ is bounded on D.

Check its the same as the traditional. Can we get the integral formula?

Show it is invertible. Hopefully using the Multi-Resolution frame.

$\mathcal{F}$ is well defined

$$\chi_{[0,1)}(x) = \chi_{[0,\frac{1}{2})}(x) + \chi_{[\frac{1}{2},1)}(x)$$

$$g(x) = g(2x) + g(2x-1)$$

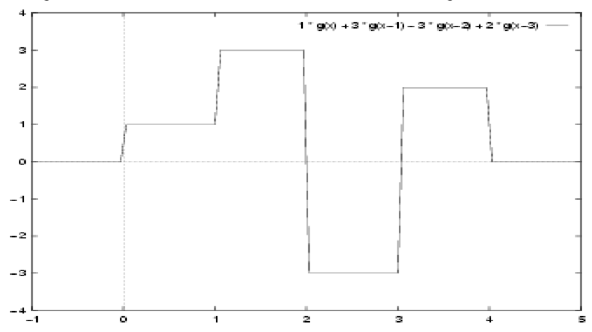
$$g(x) = \mathcal{D}_2 g(x) + \mathcal{D}_2 \mathcal{T}_{-1} g(x)$$

Applying $\mathcal{F}$ and using rules:

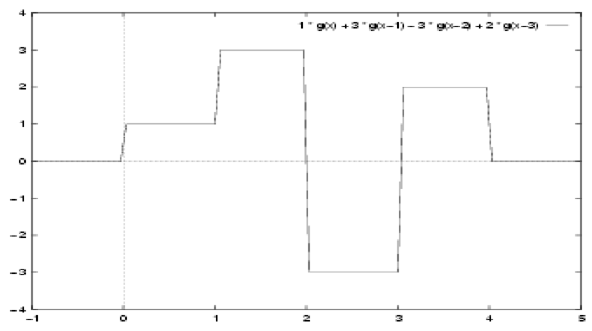
$$\mathcal{F}(g)(\omega) = \frac{1}{2}\mathcal{F}(g)\left(\frac{\omega}{2}\right) + \frac{1}{2}e^{-i\frac{\omega}{2}}\mathcal{F}(g)\left(\frac{\omega}{2}\right)$$

Checking $\mathcal{F}(g)(\omega) = \frac{1-e^{-i\omega}}{i\omega}$ makes this OK.

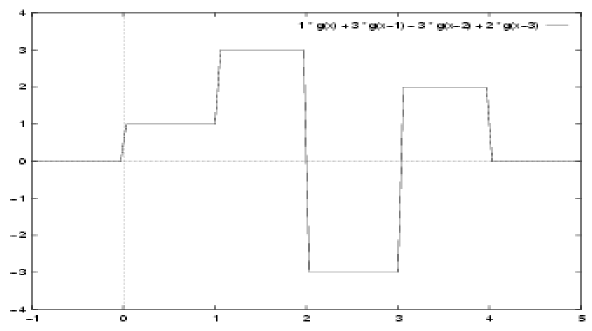
Let $f \in D$. Suppose $f \in V_J$, and $f \in V_K$.



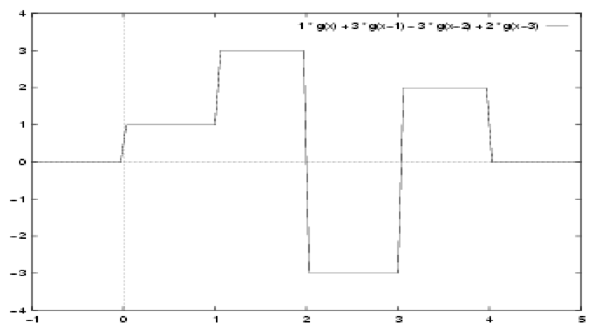
V_K



$\vdots$



$\vdots$



V_J

Showing $\mathcal{F}$ is bounded on D

Suppose:

$$f(x) = \sum a_k g(2^J x - k)$$

Then $\|f\|_2^2 = \sum |a_k|^2/2^J$. Now find $\|\mathcal{F}(f)\|_2^2$.

$$\mathcal{F}(f)(\omega) = \frac{1 - e^{-i\frac{\omega}{2^J}}}{i\omega} \sum a_k e^{-ik\frac{\omega}{2^J}}$$

$$\begin{aligned}
&\|\mathcal{F}(f)\|_2^2 \\
&= \int \left| \frac{1 - e^{-i\frac{\omega}{2^J}}}{i\omega} \right|^2 \left(\sum a_k e^{-ik\frac{\omega}{2^J}} \right) \left(\sum \overline{a_l} e^{il\frac{\omega}{2^J}} \right) d\omega \\
&= \sum \sum a_k \overline{a_l} \int \frac{2(1 - \cos\frac{\omega}{2^J})}{\omega^2} e^{-i(k-l)\frac{\omega}{2^J}} d\omega \\
&= \sum \sum a_k \overline{a_l} \frac{2\pi}{2^J} \delta_{kl} \\
&= \frac{2\pi}{2^J} \sum |a_k|^2
\end{aligned}$$

So $\mathcal{F}$ is bounded from D to $L^2(\mathbb{R})$.

Bonus: Plancherel's Theorem !

$$\|\mathcal{F}(f)\|_2^2 = 2\pi\|f\|_2^2$$

$$\Downarrow$$

$$(f, g) = 2\pi(\mathcal{F}(f), \mathcal{F}(g))$$

Review:

- We defined $\mathcal{F}$ on g.
- Translation rule defined $\mathcal{F}$ on $g(x - n)$.
- Linearity defined $\mathcal{F}$ on V_0.
- Dilation rule defined $\mathcal{F}$ on D.
- We checked $\mathcal{F}$ was well defined.
- We show $\mathcal{F}$ is bounded on D and so we can take limits to define $\mathcal{F}$ on $L^2(\mathbb{R})$.

Integral Formula

Let $f \in V_j$ for $j \geq J$ and f supported on $[-R, R]$. Then :

$$f(x) = \sum_{k=-R2^J}^{R2^J-1} a_k \chi_{[0,1)}(2^J x - k)$$

$$f(x) = \sum_{k=-R2^j}^{R2^j-1} f(\frac{k}{2^j}) \chi_{[0,1)}(2^j x - k)$$

So:

$\mathcal{F}(f)(\omega)$

$$= \quad \frac{1 - e^{-i\frac{\omega}{2^j}}}{i\omega} \sum_{k=-R2^j}^{R2^j-1} f(\frac{k}{2^j}) e^{-ik\frac{\omega}{2^j}}$$

$$= \quad \left(\frac{1 - e^{-i\frac{\omega}{2^j}}}{i\frac{\omega}{2^j}} \right) \left(\frac{1}{2^j} \sum_{k=-R2^j}^{R2^j-1} f(\frac{k}{2^j}) e^{-ik\frac{\omega}{2^j}} \right)$$

$$= \quad 1 \qquad \int_{-R}^{R} f(x) e^{-i\omega x}\, dx$$

Let $h \in L^2(\mathbb{R})$ and take $h_n \to h$ in D all supported on $[-R, R]$.

$$\begin{aligned} \mathcal{F}_R(f)(\omega) &= \int_{-R}^{R} f(x)e^{-i\omega x}\,dx \\ &= (f, \chi_{[-R,R)}e^{i\omega x}) \\ \Rightarrow \|\mathcal{F}_R(f)\|_2 &\leq \|f\|_2\sqrt{2R} \end{aligned}$$

So for h:

$$\begin{aligned} \mathcal{F}(h) &= \lim \mathcal{F}(h_n) = \lim \mathcal{F}_R(h_n) \\ &= \mathcal{F}_R(\lim h_n) = \mathcal{F}_R(h) \end{aligned}$$

If $h \in L^2(\mathbb{R})$, then $h_n = h\chi_{[-n,n)} \to h$. So in $L^2(\mathbb{R})$:

$$\mathcal{F}(h) = \lim \mathcal{F}(h_n)$$

and pointwise:

$$\begin{aligned} \lim \mathcal{F}(h_n)(\omega) &= \lim \int_{-n}^{n} h(x)e^{-i\omega x}\,dx \\ &= \int_{-\infty}^{\infty} h(x)e^{-i\omega x}\,dx \end{aligned}$$

Properties of $\mathcal{F}$

Define:

$$\mathcal{R}_n(f)(x) = e^{inx} f(x)$$

Then $\mathcal{F}$ has the following properties on D:

$$\mathcal{F}(\mathcal{T}_n(f)) = \mathcal{R}_n(\mathcal{F}(f))$$

$$\mathcal{F}(\mathcal{D}_\lambda(f)) = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}(\mathcal{F}(f))$$

$$\mathcal{F}(\mathcal{R}_n(f)) = \mathcal{T}_{-n}(\mathcal{F}(f))$$

All of the form $\mathcal{F}\mathcal{A} = \mathcal{B}\mathcal{F}$, with $\mathcal{A}, \mathcal{B}$ bounded and linear.

It is easy to extend these to all of $L^2(\mathbb{R})$ by limits.

Inverse Transform

1. $\mathcal{G}\left(g\right)\left(\omega\right) = \frac{1}{2\pi}\frac{1-e^{i\omega}}{i\omega}$
2. $\mathcal{G}\left(\mathcal{T}_n f\right)\left(\omega\right) = e^{i\omega n}\mathcal{G}\left(f\right)\left(\omega\right)$ for $n \in \mathbb{Z}$.
3. $\mathcal{G}$ is linear.
4. $\mathcal{G}\left(\mathcal{D}_\lambda f\right)\left(\omega\right) = \frac{1}{|\lambda|}\mathcal{G}\left(f\right)\left(\frac{\omega}{\lambda}\right)$ where $\lambda = 2^n, n \in \mathbb{Z}$.

Then:

- $\mathcal{G}$ is well defined.
- $\mathcal{G}$ is bounded with norm $1/\sqrt{2\pi}$.
- $\mathcal{G}$ has integral formula $\frac{1}{2\pi}\int f(x)e^{i\omega x}\,dx$.
- $\mathcal{G}$ has properties:

$$\mathcal{G}(\mathcal{T}_n(f)) = \mathcal{R}_{-n}(\mathcal{G}(f))$$

$$\mathcal{G}(\mathcal{D}_\lambda(f)) = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}(\mathcal{G}(f))$$

$$\mathcal{G}(\mathcal{R}_n(f)) = \mathcal{T}_n(\mathcal{G}(f))$$

We look at $\mathcal{I} = \mathcal{G} \circ \mathcal{F}$:

1. $\mathcal{I}\,(g)\,(\omega) = ???$
2. $\mathcal{I}\,(\mathcal{T}_n f) = \mathcal{T}_n\,(\mathcal{I}\,(f))$ for $n \in \mathbb{Z}$.
3. $\mathcal{I}$ is linear.
4. $\mathcal{I}\,(\mathcal{D}_\lambda f) = \mathcal{D}_\lambda\,(\mathcal{I}\,(f))$ where $\lambda = 2^n, n \in \mathbb{Z}$.

To find $\mathcal{I}(g) = \mathcal{G} \circ \mathcal{F}(g)$ use the formula:

$$\begin{aligned} &\int \frac{1 - e^{-i\omega}}{i\omega} e^{i\omega x}\, d\omega \\ &= \int \frac{e^{i\omega x} - e^{i\omega(x-1)}}{i\omega}\, d\omega \\ &= \int \frac{\sin \omega x - \sin \omega(x-1)}{\omega}\, d\omega \\ &= (\operatorname{sign}(x) - \operatorname{sign}(x-1)) \int \frac{\sin \omega}{\omega}\, d\omega \\ &= g(x) 2\pi \end{aligned}$$

Summing up

- Easier than Schwartz Class ?
- Hands on.
- $\mathbb{R}^n$ should be easy.
- Well defined weakens:

$$\mathcal{F}(g)(\omega) = \frac{1 - e^{-i\omega}}{i\omega}$$

to:

$$\mathcal{F}(g)(0) = 1 \ , \ \mathcal{F}(g) \, \text{ctns. at } 0$$