Self Affine Tiles and Dilation Equations

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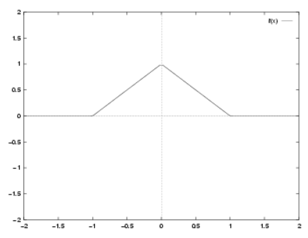
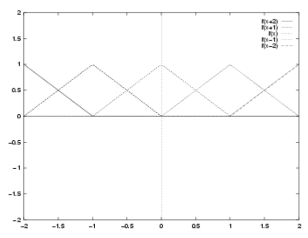
Multi-Resolution Analysis

To approximate something we:

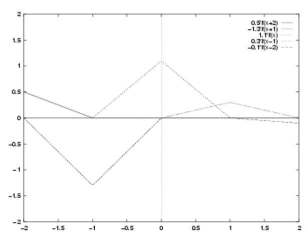
1. Take a basis function g.



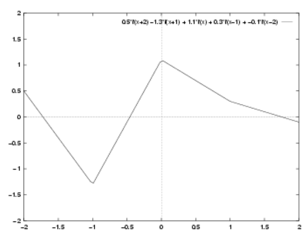
2. Translate to nodes.



3. Multiply by coefficients.



4. Sum the results.



To improve, try moving nodes closer together.

Definition of a Multiresolution Analysis:

1. A set V_0:

$$V_0 = \operatorname{span}\{g(x-n) : n \in \mathbb{Z}\}$$

2. The $g(x-n)$ should be orthogonal.
3. Add multiple resolutions:

$$f(x) \in V_k \iff f(2x) \in V_{k+1}$$

4. They should be increasing, ie. $V_k \subset V_{k+1}$.
5. The union should be dense:

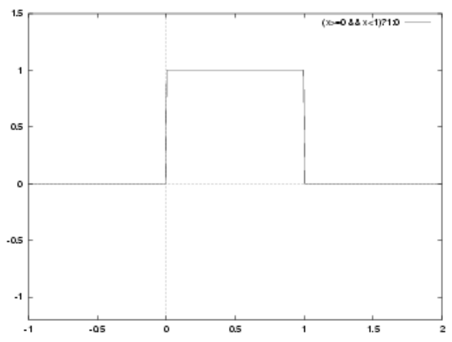
$$\overline{\bigcup_{j=-\infty}^{+\infty} V_j} = L^2(\mathbb{R})$$

6. The intersection should be zero:

$$\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$

Examples

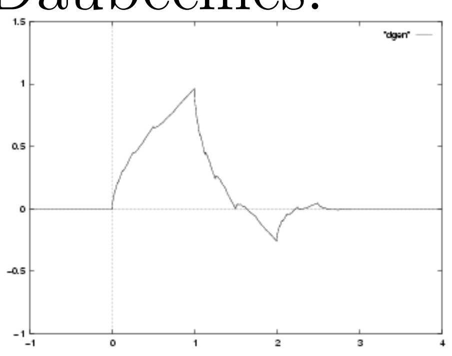
1. Haar: $g(x) = \chi_{[0,1)}$



   $g(x) = g(2x) + g(2x-1)$.

2. Shannon: $g(x) = \frac{\sin \pi x}{\pi x}$
   In $\widehat{V}_j$ if supported on $[-2^j\pi, 2^j\pi]$.
   $g(x) = g(2x) + \sum \frac{(-1)^n 2}{\pi n} g(2x-2n)$.

3. Daubechies:



   $g(x) = \frac{1}{4}[(1+\sqrt{3})g(2x) + (3+\sqrt{3})g(2x-1) + (3-\sqrt{3})g(2x-2) + (1-\sqrt{3})g(2x-3)]$.

Haar on $\mathbb{R}^n$

A simple generalisation to $\mathbb{R}^2$ might take the function χ_Q where $Q = [0, 1) \times [0, 1)$. This works in our definition of MRA by just replacing $\mathbb{R}$ with $\mathbb{R}^2$ and $\mathbb{Z}$ with $\mathbb{R}^2$.

This is really a product of two one dimensional MRAs. Do more interesting structures exist on $\mathbb{R}^n$?

Instead of $\mathbb{Z}^n$ use a full rank lattice Γ.

Instead of dilation by two use a mairix A whose eigenvalues have norm bigger than one and which leaves Γ fixed.

Multiresolution Analysis of scale A on lattice Γ:

1. V_0 s.t.:

$$V_0 = \operatorname{span}\{g(x-\gamma) : \gamma \in \Gamma\}$$

2. The $g(x-\gamma)$ should be orthogonal.
3. Multiple resolutions:

$$f(x) \in V_k \iff f(Ax) \in V_{k+1}$$

4. Increasing: $V_k \subset V_{k+1}$.
5. Dense Union:

$$\overline{\bigcup_{j=-\infty}^{+\infty}} V_j = L^2(\mathbb{R})$$

6. Zero intersection:

$$\bigcap_{j=-\infty}^{+\infty} V_j = \{0\}$$

Haar Bases and Self-Affine Tiles

Gröchenig and Madych (1992):

Theorem 1 *If Q is a bounded and measurable set, then χ_Q generates a MRA iff:*

1. *$Q \cap (Q + k)$ has measure zero for $k \in \mathbb{Z}^n \setminus \{0\}$.*
2. *There is a collection of distinct coset representatives of $\mathbb{Z}^n/A\mathbb{Z}^n$ such that:*

$$AQ = \cup_{i=1}^{q}(k_i + Q).$$

3. *Q tiles $\mathbb{R}^n$ when translated by $\mathbb{Z}^n$.*

Consequently: $|Q| = 1$ and $q = |\det(A)|$.

Digit Sets

Consider iterating:

$$Q = \cup_{i=1}^{q} A^{-1}(k_i + Q)$$

$$Q = \cup_{i=1}^{q} \cup_{j=1}^{q} A^{-1}k_i + A^{-2}k_j + A^{-2}Q$$

But $A^{-n}Q \to 0$, so:

$$Q = \left\{ \sum_{j=1}^{\infty} A^{-j}\epsilon_j : \epsilon_j \in \{k_1, \ldots, k_q\} \right\}$$

This is like a base A expansion of the points in Q, and so $\{k_1, \ldots, k_q\}$ is called the digit set.

Examples

1. On $\mathbb{R}$ and taking dilation by 2, we can take $\{0, 1\}$ as coset reps of $\mathbb{Z}/2\mathbb{Z}$. This leads to:

$$\sum_{j=1}^{\infty} \frac{1}{2^j}\epsilon_j \text{ where } \epsilon_j \in \{0, 1\}$$

   ie. the binary expansion of points in $[0, 1]$.

2. By translation we can ensure that 0 is always in the digit set. For instance $\{3, 4\}$ leads to $Q = [3, 4] = 3 + [0, 1]$. Other representatives lead to stretched sets $\{0, 5\} \Rightarrow [0, 5]$.

Gröchenig and Madych (1992):

Theorem 2 *Given* $\{k_1, \ldots, k_q\}$ *distinct coset representatives of* $\mathbb{Z}^n/A\mathbb{Z}^n$*, and* Q *produced from these digits, then TFAE:*

1. χ_Q *generates a MRA,*
2. $|Q| = 1$*,*
3. $k + Q$ *are essentially disjoint for* $k \in \mathbb{Z}^n$*.*

(Also includes three technical conditions).

To produce a Haar like MRA of scale A we must select a digit set which generates a set Q of measure 1. Is this always possible?

It was shown that two necessary conditions were that the digits form a complete set of coset representatives and that $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}$.

This was shown to be sufficient when $n = 1$ or when $\det(A)$ was prime (characteristic poly of A irreducable).

Suitable sets of digits were definitely shown to exist when $\det(A) \geq n + 1$ and when $n = 1, 2, 3$.

Unfortunately this missed the most interesting case for the wavelets people. When $\det(A) = 2$ you only need one wavelet, and this case wasn't covered.

In 1994 Lagarias and Wang (almost) proved the following:

Theorem 3 *If f, the characteristic polynomial of A, is irreducable with $|f(0)| = 2$ then A has a primitive complete digit set iff $\mathbb{Z}[1, \theta, \ldots, \theta^{n-1}]$ has class number 1.*

In 1997 Potiopa found an example which didn't have class number 1.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

My Aim: Find characteristic functions that satisfy dilation equations.

What do we already know?

1. $\chi_{[0,1)}$ satisfies $f(x) = \sum_{k=0}^{n-1} f(nx-k)$ for $n \in \mathbb{N}$, $n > 0$.
2. Cantor's middle third set satisfies $f(x) = f(3x) + f(3x-2)$.
3. $\chi_{\mathbb{R}}(x) = \sum c_k \chi_{\mathbb{R}}(2x-k)$ for any c_k summing to one.
4. $\chi_{[0,1)\cup[2,3)}$ satisfies a scale 2 equation $c_n = 1, 1, -1, -1, 2, 2, -2, -2, \ldots$.
5. Self affine tiles which generate MRAs satisfy dilation equations with $c_n \in \{0, 1\}$ due to orthogonality.

Left/Right Hand End

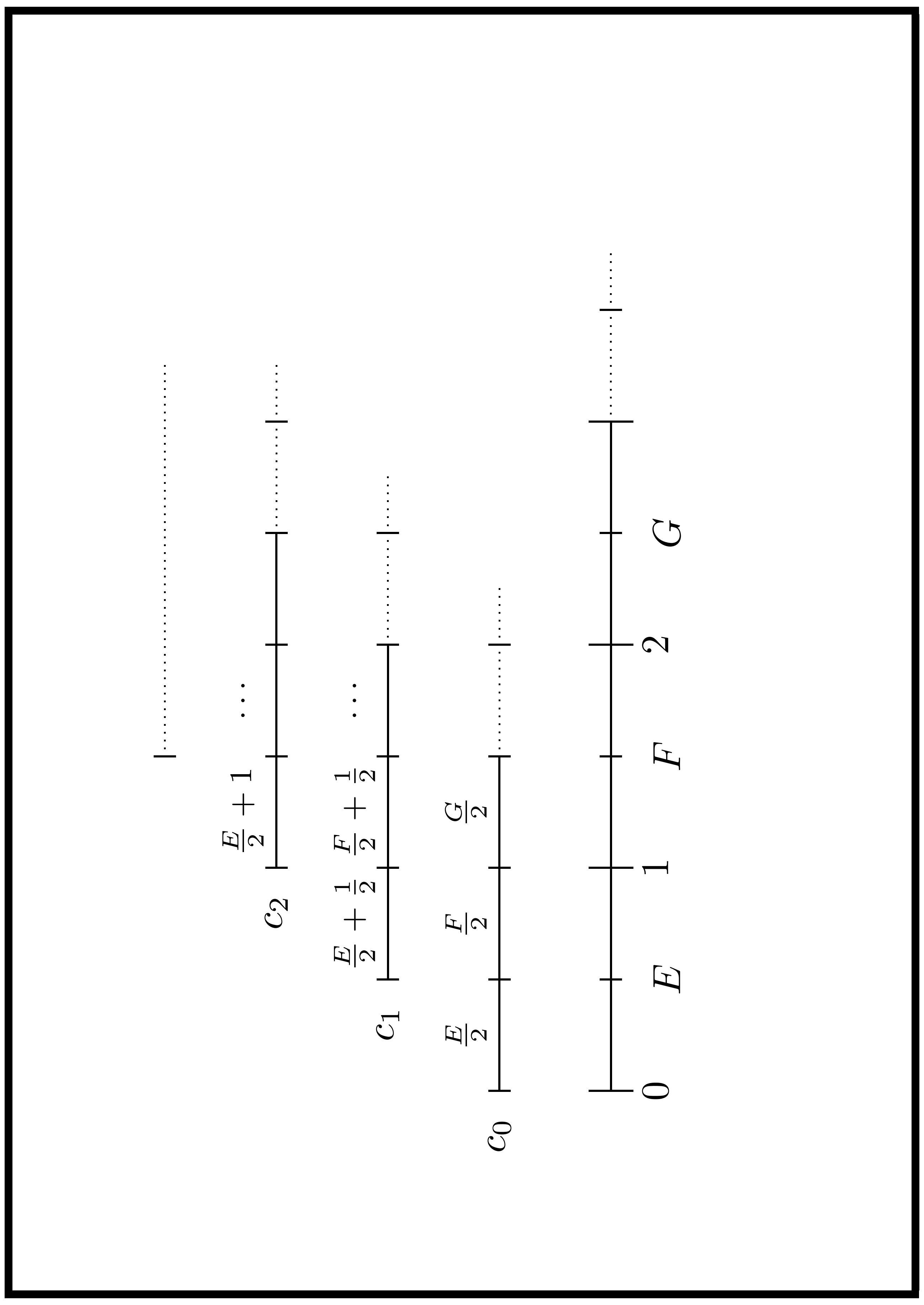
Assume: Compact support & finite nonzero c_n.

Lemma 4 *Suppose* $g(x) = \sum d_k g(2x - k)$, *and only finitely many of the* d_k *are non-zero. Then we can find* l *so that if* $f(x) = g(x - l)$ *we find:*

$$f(x) = \sum c_k f(2x - k),$$

$c_0 \neq 0$, $c_k = 0$ *when* $k < 0$ *and* $c_k = d_{k-l}$.

Lemma 5 *If* f *is compactly supported and satisfies a dilation equation* $f(x) = \sum c_k f(2x - k)$, *where* $c_0 \neq 0$ *and* $c_k = 0$ *when* $k < 0$, *then* f *is zero almost everywhere in* $(-\infty, 0)$.


c_0
$\frac{E}{2}$
$\frac{F}{2}$
$\frac{G}{2}$
c_1
$\frac{E}{2}+\frac{1}{2}$
$\frac{F}{2}+\frac{1}{2}$
c_2
$\frac{E}{2}+1$
$\cdots$
$\cdots$
0
E
1
F
2
G

Theorem 6 *If S is bounded and satisifies a dilation equation $\chi_S(x) = \sum c_k \chi_S(2x-k)$ a.e., where $c_0 \neq 0$ and $c_k = 0$ when $k < 0$, then either:*

- *S is of measure zero or,*
- *$c_0 = 1$, the rest of the c_k are integers with $|c_k| \leq 2^k$ and E has non-zero measure in both $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$.*

Problem: For scale 2 it looks like E must be the whole interval. For the moment we'll assume it.

Theorem 7 *The map from functions which are constant on $[n, n+1)$ to the polynomials given by:*

$$f(x) = \sum_r a_r \chi_{[r,r+1)}(x) \mapsto \sum_r a_r x^r = P_f(x),$$

is a linear bijection, transforming the following operations in the following way:

$$\begin{aligned}
(\alpha f + \beta g)(x) &\mapsto \alpha P_f(x) + \beta P_g(x), \\
f\left(\frac{x}{n}\right) &\mapsto \frac{x^n - 1}{x - 1} P_f(x^n), \\
f(x-k) &\mapsto x^k P_f(x), \\
\sum_k c_k f(x-k) &\mapsto P_f(x) Q(x),
\end{aligned}$$

where $Q(x) = \sum c_k x^k$.

Our dilation equation becomes:

$$P(x)Q(x) = P(x^2)(x + 1).$$

Messing with polynomials and roots gives:

1. $R(x)Q(x) = R(x^2)$ iff when r a root of R of order p then r^2 is a root of R of order atleast p.
2. $P(x) = R(x)/(x - 1)$ where $R(1) = 0$.
3. All of P's roots are either 0 or a root of unity.
4. If P's coefficients are real then P is palendromic or anti-palendromic.

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P = 1, Q = 1 1
P = 1 1, Q = 1 0 1
P = 1 1 1, Q = 1 0 0 1
P = 1 0 1 0 1, Q = 1 1 -1 -1 1 1
P = 1 1 1 1, Q = 1 0 0 0 1
P = 1 1 1 1 1, Q = 1 0 0 0 0 1
P = 1 0 0 1 0 0 1, Q = 1 1 0 -1 -1 0 1 1
P = 1 1 1 1 1 1, Q = 1 0 0 0 0 0 1
P = 1 1 1 1 1 1 1, Q = 1 0 0 0 0 0 0 1
P = 1 0 0 0 1 0 0 0 1, Q = 1 1 0 0 -1 -1 0 0 1 1
P = 1 0 1 0 1 0 1 0 1, Q = 1 1 -1 -1 1 1 -1 -1 1 1
P = 1 1 0 1 1 0 1 1, Q = 1 0 1 -1 0 -1 1 0 1
P = 1 1 1 1 1 1 1 1, Q = 1 0 0 0 0 0 0 0 1
P = 1 1 1 1 1 1 1 1 1, Q = 1 0 0 0 0 0 0 0 0 1
P = 1 0 0 0 0 1 0 0 0 0 1, Q = 1 1 0 0 0 -1 -1 0 0 0 1 1
P = 1 1 0 0 1 1 0 0 1 1, Q = 1 0 1 0 -1 0 -1 0 1 0 1
P = 1 1 1 0 1 1 1 0 1 1 1, Q = 1 0 0 1 -1 0 0 -1 1 0 0 1
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