$L^2(\mathbb{R})$ Solutions of Dilation Equations and Fourier-like Transforms

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Abstract

We state a novel construction of the Fourier transform on $L^2(\mathbb{R})$ based on translation and dilation properties which makes use of the multiresolution analysis structure commonly used in the design of wavelets. We examine the conditions imposed by variants of these translation and dilation properties. This allows other characterisations of the Fourier transform to be given, and operators which have similar properties are classified. This is achieved by examining the solution space of various dilation equations, in particular we show that the $L^2(\mathbb{R})$ solutions of $f(x) = f(2x) + f(2x-1)$ are in direct correspondence with $L^2(\pm[1,2))$.

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1 Introduction and Notation

Two scale dilation equations or *refinement equations* (such as $f(x) = f(2x) + f(2x-1)$) have been studied in detail recently. This is because of applications to the construction of wavelets [3], and approximation of curves and surfaces in computer graphics [1]. There are many interesting works concerning dilation equations; see the references for a small sample. In the context of this paper, where we are concerned with existence and uniqueness of solutions to dilation equations, [4] provides a good background. However these works concentrate either on the $L^1(\mathbb{R})$ or compactly supported $L^p(\mathbb{R})$ solutions of dilation equations. This ensures the continuity of the Fourier transform of the solutions, making certain uniqueness arguments possible.

Mallat's *multiresolution analysis* structure (eg. [8]), which is used by those constructing wavelets, is closely linked with two scale dilation equations. This structure is used in the construction mentioned in Section 2, which focuses on the interaction between dilation, translation and the Fourier transform uses. The construction is interesting because it is straightforward (for example cf. [10]) and it also highlights the possibility of constructing other operators which have translation and dilation properties similar to those of the Fourier transform.

In Section 3 we eliminate these other operators by considering several two scale dilation equations simultaneously. This provides a another characterisation of the Fourier transform. The argument shows that in $L^2(\mathbb{R})$ solutions of two scale dilation equations are not unique.

Section 4 looks at the idea of a "maximal" solution to a dilation equation.

This is a solution from which all other solutions can be derived. We show that the $L^2(\mathbb{R})$ solutions of a most basic dilation equation: $f(x) = f(2x) + f(2x - 1)$ are actually in correspondence with $L^2(\pm[1, 2))$. This is used to classify all operators which behave in a similar manner to the Fourier transform with respect to shifts (translations by integer amounts) and dilation by two.

Finally, Theorem 10 of Section 5 summarises the operator results of the previous sections in terms of operators which commute with shifts and dilations by various scales.

We make the following definitions. Let $\{V_j\}_{j\in\mathbb{Z}}$ be the sets of the multiresolution analysis generated by $\chi_{[0,1)}$, the characteristic function of $[0, 1)$. Explicitly:

$$V_j = \operatorname{span}\{\chi_{[0,1)}(2^j x - r) : r \in \mathbb{Z}\},$$

or V_j contains simple functions constant on $2^{-j}[r, r + 1)$. Note that $V_j \subset V_{j+1}$ and $f(x) \in V_j$ iff $f(2x) \in V_{j+1}$. Let $D := \bigcup V_j$ — this set is dense in $L^2(\mathbb{R})$.

We use $\hat{f}$ or $\mathcal{F}f$ to denote the Fourier transform of f. We also use the following linear operators:

Translation: $(\mathcal{T}_\alpha f)(x) := f(x + \alpha)$ for any $\alpha \in \mathbb{R}$,

Rotation: $(\mathcal{R}_\alpha f)(x) := e^{i\alpha x} f(x)$ for any $\alpha \in \mathbb{R}$,

Dilation: $(\mathcal{D}_\lambda f)(x) := f(\lambda x)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$.

2 Construction of the Fourier Transform on $L^2(\mathbb{R})$

In [6, 7] it was shown that we could construct the Fourier transform on $L^2(\mathbb{R})$ using the following definition in the spirit of multiresolution analysis.

Definition 1. We define an operator on D using the following rules:

(1.1) $\mathcal{F} : D \to L^2(\mathbb{R})$ is linear,

(1.2) $\mathcal{F}\mathcal{T}_n f = \mathcal{R}_n \mathcal{F} f$ for all $n \in \mathbb{Z}$ and $f \in D$,

(1.3) $\mathcal{F}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{F} f$ for all $\lambda \in 2^{\mathbb{Z}}$ and $f \in D$,

(1.4) $\mathcal{F}(\chi_{[0,1)}) = \frac{1-e^{-i\omega}}{i\omega}$.

The last rule (1.4) allows us to define $\mathcal{F}$ on the generating function of V_0. The "translation property" (1.2) allows us to define $\mathcal{F}$ on the shifts of the generating function. The linearity (1.1) allows us to define $\mathcal{F}$ on all of V_0 and the "dilation property" (1.3) allows us to define $\mathcal{F}$ on each V_j and so on all of D.

The expected properties of $\mathcal{F}$, such as continuity and invertibility, can be now be derived in a manner with a feel of multiresolution analysis. This definition also works well on $L^2(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ with little modification.

While checking this definition led to a well defined $\mathcal{F}$ we had to check that the dilation equation:

$$\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1),$$

was respected by $\mathcal{F}$. By application of the rules this led to:

$$\frac{1-e^{-i\omega}}{i\omega} = \left(\frac{1+e^{-i\frac{\omega}{2}}}{2}\right)\left(\frac{1-e^{-i\frac{\omega}{2}}}{i\frac{\omega}{2}}\right)$$

which is clearly true. Iterating this we get:

$$(\mathcal{F}\chi_{[0,1)})(\omega) = p_2(\frac{\omega}{2})p_2(\frac{\omega}{4})\dots p_2(\frac{\omega}{2^m})(\mathcal{F}\chi_{[0,1)})(\frac{\omega}{2^m})$$

where $p_2(\omega) = (1+e^{i\omega})/2$. By forming the implied infinite product we see that the following is actually sufficient to define $\mathcal{F}$ (on D).

Definition 2. Define $\mathcal{F}$ using (1.1),(1.2) and (1.3), and also:

(2.4) $\mathcal{F}(\chi_{[0,1)})$ is continuous at zero with value 1.

This second definition is also “natural” in the sense that we expect the Fourier transform of L^1 functions to be continuous. The complete details of all these constructions can be found in Chapter 3 of [6].

3 Stronger Dilation Rules

If we have a linear operator $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with the dilation property for scales $2^{\mathbb{Z}}$ (1.3) and the translation property for integers (1.2) then we have:

$$(\mathcal{A}\chi_{[0,1)})(\omega) = p_2(\frac{\omega}{2})(\mathcal{A}\chi_{[0,1)})(\frac{\omega}{2}).$$

It is easy to verify that some functions in $L^2(\mathbb{R})$, other than $\mathcal{F}\chi_{[0,1)}$, are candidates for $\mathcal{A}\chi_{[0,1)}$. Two examples are $-i\,\mathrm{sign}(\omega)\hat{\chi}_{[0,1)}(\omega)$ and $\hat{\chi}_{[0,1)}(\omega)\chi_{2^{\mathbb{Z}}E}(\omega)$ where $E \subset [1,2)$. By using these as $\mathcal{A}\chi_{[0,1)}$ and proceeding as in Section 2 we could attempt to produce Fourier-like transforms.

We eliminate these other solutions by allowing dilations by other scales and arrive at another classification of the Fourier transform. We begin by making two observations.

First, $\chi_{[0,1)}$ satisfies the dilation equation $f(x) = f(2x) + f(2x-1)$. This produced the relation between $\mathcal{A}\chi_{[0,1)}$ at ω and at $\frac{\omega}{2}$. Other dilation equations which $\chi_{[0,1)}$ satisfies produce other relations. In particular it satisfies all of these "dilation equations".

$$\begin{aligned} f(x) &= f(nx) + f(nx-1) + \ldots + f(nx-n+1) \qquad n \in \mathbb{N} \\ f(x) &= f(1-x) \end{aligned}$$

If $\mathcal{A}$ has the dilation property for the scale n and -1 respectively then $\mathcal{A}\chi_{[0,1)}$ satisfies:

$$\begin{aligned} (\mathcal{A}\chi_{[0,1)})(\omega) &= p_n(\frac{\omega}{n})(\mathcal{A}\chi_{[0,1)})(\frac{\omega}{n}), \\ (\mathcal{A}\chi_{[0,1)})(\omega) &= e^{-i\omega}(\mathcal{A}\chi_{[0,1)})(-\omega). \end{aligned}$$

Here $p_n(\omega) = \frac{1}{n}\frac{1-e^{-i\omega}}{1-e^{-i\omega/n}}$, which is an analytic function with period 2π.

Second, suppose we know a function $\tilde{f}$ on $[0,\epsilon)$ which is a solution to an equation like $\tilde{f}(\omega) = p(\frac{\omega}{\alpha})\tilde{f}(\frac{\omega}{\alpha})$ where $\alpha > 1$. We can determine $\tilde{f}$ on all of $\mathbb{R}^+$. Similarly if we know $\tilde{f}$ on $\mathbb{R}^+$ and $\tilde{f}$ is a solution of $\tilde{f}(\omega) = e^{-i\omega}\tilde{f}(-\omega)$ then we can determine $\tilde{f}$ on all of $\mathbb{R}$.

The following theorem uses all the dilation equations above, so we assume the dilation property for all necessary scales.

Theorem 3. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear transform with the translation property (1.2) and the dilation property for scales $n \in \mathbb{Z} \setminus \{0\}$:*

(3.3) $\mathcal{A}\mathcal{D}_\lambda f = \frac{1}{|\lambda|}\mathcal{D}_{\frac{1}{\lambda}}\mathcal{A}f$ *for all* $\lambda \in \mathbb{Z} \setminus \{0\}$ *and* $f \in D$.

Then $\mathcal{A}$ is a constant multiple of the Fourier transform.

Proof. Let $\tilde{f} := \mathcal{A}\chi_{[0,1)}$. Then we know that $\tilde{f}$ satisfies:

$$\tilde{f} = \mathcal{D}_{\frac{1}{n}} p_n \tilde{f} \qquad n \in \mathbb{N}.$$

We also know that $\tilde{f}_0 := \mathcal{F}\chi_{[0,1)}$ satisfies the same relations. Noting that $\tilde{f}_0$ is not zero in the interval $[0, \pi]$, we can divide by it. Likewise, none of the p_n are zero on $[0, \pi]$, so we cancel the p_n:

$$\frac{\tilde{f}}{\tilde{f}_0} = \frac{\mathcal{D}_{\frac{1}{n}} p_n \tilde{f}}{\mathcal{D}_{\frac{1}{n}} p_n \tilde{f}_0} = \frac{\mathcal{D}_{\frac{1}{n}} \tilde{f}}{\mathcal{D}_{\frac{1}{n}} \tilde{f}_0} = \mathcal{D}_{\frac{1}{n}} \frac{\tilde{f}}{\tilde{f}_0}.$$

Writing $g := \tilde{f}/\tilde{f}_0$ on $[0, \pi]$ we get $g = \mathcal{D}_{\frac{1}{n}} g$. Iterating this for values of n, m:

$$\mathcal{D}_{\frac{n}{m}} g = \mathcal{D}_{\frac{1}{m}} \mathcal{D}_n g = \mathcal{D}_{\frac{1}{m}} g = g.$$

So for $\alpha \in \mathbb{Q}$ we have $g = \mathcal{D}_\alpha g$, while both sides are evaluated in $[0, \pi]$.

Now consider $G(\omega) := \int_0^\omega g(t)\,dt$. As $\tilde{f}$ is in $L^2(\mathbb{R})$, g is in $L^1([0,\pi])$. Thus G is continuous. However for rational $\alpha \in (0, \pi]$:

$$G(\alpha) = \int_0^\alpha g(t)\,dt = \int_0^\alpha g(\frac{t}{\alpha})\,dt = \alpha \int_0^1 g(s)\,ds$$

So $G(\alpha) = \alpha G(1)$. But G is continuous, so $G(\omega) = \omega G(1)$ for $\omega \in [0, \pi]$.

From the definition of G we have $G' = g$ almost everywhere, thus $g(\omega) = G(1)$ at almost every point in $[0, \pi]$. This means that $\tilde{f}$ is a constant multiple of $\tilde{f}_0$ on $[0, \pi]$. However, functions satisfying the dilation relations are determined by their value of $[0, \pi]$. Thus, $\tilde{f}$ and $\tilde{f}_0$ must be constant multiples of one another almost everywhere.

To complete the proof we apply the translation property & linearity, dilation property and boundedness to get $\mathcal{A} = c\mathcal{F}$ on V_0, D and $L^2(\mathbb{R})$ respectively. ■

Suppose $\mathcal{A}$ has the dilation property for scales in $S \subset \mathbb{Z} \setminus \{0\}$. We see that what was used in the above proof was that $S^\times$ (the multiplicative group generated by S) was dense in $\mathbb{R}$. Much smaller sets than all of $\mathbb{Z} \setminus \{0\}$ do this, for example $S = \{-1, 2, 3\}$.

This proof can be used to prove the uniqueness of $L^2(\mathbb{R})$ solutions to certain systems of dilation equations, providing a reasonably well behaved solution exists.

4 Maximal Solutions to Dilation Equations

We now consider a slightly different question, where we allow dilations of only one scale. Suppose we take a bounded linear operator $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with the translation property for integers and the dilation property for 2. What choices do we have for $\mathcal{A}\chi_{[0,1)}$? This question is related to the problem of finding all the solutions to $f(x) = f(2x) + f(2x-1)$, or $\tilde{f}(\omega) = p_2(\frac{\omega}{2})\tilde{f}(\frac{\omega}{2})$.

Initially we consider the second form of this problem, in a completely pointwise manner. Define $F(p)$ as follows:

$$F(p) := \left\{ \tilde{f} : \mathbb{R} \to \mathbb{C} : \tilde{f}(\omega) = p(\frac{\omega}{2})\tilde{f}(\frac{\omega}{2}) \right\}$$

We note that if $\tilde{f} \in F(p)$ and if π is chosen so that $\pi(\omega) = \pi(2\omega)$ for all $\omega \in \mathbb{R}$ then $\pi\tilde{f}$ is also in $F(p)$.

A converse to this is: We can find $m \in F(p)$ so that for any $\tilde{f}$ in $F(p)$ there is a π such that $f = m\pi$ and $\pi(\omega) = \pi(2\omega)$. The proof is in the form of the two following lemmas.

Lemma 4. *We can find m in $F(p)$ such that $m(\omega) = 0$ implies $\tilde{f}(\omega) = 0$ for all $\tilde{f} \in F(p)$.*

Proof. For each $y \in \pm[1, 2)$ we examine the following set:

$$\{n \in \mathbb{Z} : p(2^n y) = 0\}.$$

If this set has no lower bound we set $m(2^n y) = 0$ for all $n \in \mathbb{Z}$. If it has a lower bound l then we set $m(2^l y) = 1$ or if it is empty we set $m(y) = 1$ and then use:

$$\begin{aligned} m(2^n y) &= \frac{m(2^{n+1}y)}{p(2^n y)}, \\ m(2^n y) &= p(2^{n-1}y)m(2^{n-1}y) \end{aligned}$$

to extend m to the set $y2^{\mathbb{Z}}$. We do not have problems dividing by zero because of where we have chosen the value of m.

It remains to define m at 0, where we want $m(0) = p(0)m(0)$, so we set $m(0) = 1$ if $p(0) = 1$ and $m(0) = 0$ otherwise.

By construction $m \in F(p)$, and by checking the various cases we can show that $m(\omega) = 0$ implies $\tilde{f}(\omega) = 0$. ■

Lemma 5. *Suppose we have $m \in F(p)$ as in Lemma 4. Then for each $\tilde{f} \in F(p)$ we can find a $\pi : \mathbb{R} \to \mathbb{C}$ so that $\tilde{f} = m\pi$ and $\pi(\omega) = \pi(2\omega)$.*

Proof. We know that is it "safe" to divide $\tilde{f}$ by m from Lemma 4. Based on this we define π by:

$$\pi(\omega) = \frac{\tilde{f}(\omega)}{m(\omega)} \text{ or } \frac{\tilde{f}(\omega/2)}{m(\omega/2)} \text{ or } \frac{\tilde{f}(\omega/4)}{m(\omega/4)} \text{ or } \frac{\tilde{f}(\omega/8)}{m(\omega/8)} \text{ or } \dots \text{ or } 0$$

depending on which one is the first to have $m(\omega/2^n) \neq 0$ for $n = 0, 1, 2, \ldots$, or if they are all zero we set $\pi(\omega) = 0$.

First we check if $\tilde{f} = \pi m$. If $m(\omega) \neq 0$ then $\tilde{f}(\omega) = \pi(\omega)m(\omega)$ by π's definition, and if $m(\omega) = 0$ then $\tilde{f}(\omega) = 0$ so the value of $\pi(\omega)$ doesn't matter.

Now we have to check if $\pi(\omega) = \pi(2\omega)$. First consider the case $m(2\omega) \neq 0$.

Then $p(\omega) \neq 0$ so:

$$\pi(2\omega) = \frac{f(2\omega)}{m(2\omega)} = \frac{f(\omega)p(\omega)}{m(\omega)p(\omega)} = \frac{f(\omega)}{m(\omega)} = \pi(\omega),$$

as $m(\omega)$ also cannot be zero. On the other hand if $m(2\omega) = 0$ then:

- either $m(2\omega/2^n) = 0$ for all $n = 0, 1, 2, 3, \ldots$, which means $\pi(2\omega) = 0$ and $\pi(\omega) = 0$, as required,
- or $\pi(2\omega) = \frac{f(\omega/2^n)}{m(\omega/2^n)}$, and $\pi(\omega)$ is the same, also as required.

■

We can now apply the above lemma to give the following result regarding dilation equations.

Theorem 6. *Given a finite dilation equation:*

$$f(x) = \sum c_n f(2x - n),$$

we can find a function $m(\omega)$ with the following property: given g a solution of the dilation equation whose Fourier transform $\hat{g}$ converges almost everywhere then $\hat{g} = \pi m$ almost everywhere and $\pi(\omega) = \pi(2\omega)$.

Proof. We apply our lemma to $F(p)$ where $p(\omega) = \frac{1}{2}\sum c_n e^{-in\omega}$. The only complication is that $\hat{g}$ may fail to satisfy the relation $\hat{g}(\omega) = p(\frac{\omega}{2})\hat{g}(\frac{\omega}{2})$ on a set of measure zero. To get around this we can redefine $\hat{g}$ on a countable union of sets of measure zero so that it is in $F(p)$. ■

We have constructed an example of a "maximal" m. This example need not be well behaved — it is not obvious that it is even measurable. However, this maximal function is not unique and there are some simple sufficient conditions for maximality. For instance, if we have $\tilde{f} \in F(p)$ with $\tilde{f}$ non-zero on $(-\epsilon, \epsilon)$ then it is easy to show that $\tilde{f}$ is maximal.

Similarly, if p is continuous at zero and $\tilde{f} \in F(p)$ is analytic then either $\tilde{f}$ is identically zero or $\tilde{f}$ is maximal. This shows the $L^1(\mathbb{R})$ solutions with $\sum c_n = 2$ (discussed in [4]) have maximal Fourier transforms, as they are analytic and p is a trigonometric polynomial.

We are now in a position to prove two quite interesting results.

Theorem 7. *The $L^2(\mathbb{R})$ solutions of:*

$$f(x) = f(2x) + f(2x-1)$$

are in a natural one-to-one correspondence with the functions in $L^2(\pm[1,2))$.

Proof. We classify the solutions of the Fourier transform of the dilation equation, and use the fact that the Fourier transform is bijective. We observe $\hat{\chi}_{[0,1)}$ is maximal so any solution of the transformed equation is of the form:

$$\hat{g} = \pi \hat{\chi}_{[0,1)}$$

with $\pi(\omega) = \pi(2\omega)$. We show that $\pi \in L^2(\pm[1,2))$ iff $\hat{g}$ is in $L^2(\mathbb{R})$.

First suppose $\hat{g} \in L^2(\mathbb{R})$. Note that $|\hat{\chi}_{[0,1)}| > 0.1$ on $[1,2]$ so:

$$\infty > \int_1^2 |\hat{g}(\omega)|^2 \, d\omega = \int_1^2 |\pi(\omega)\hat{\chi}_{[0,1)}(\omega)|^2 \, d\omega > (0.1)^2 \int_1^2 |\pi(\omega)|^2 \, d\omega,$$

So $\pi \in L^2([1,2))$. Similarly we show $\pi \in L^2(-[1,2))$.

Conversely, suppose π is in $L^2(\pm[1,2))$. We use the fact that $\hat{\chi}_{[0,1)}$ is bounded near zero and $\hat{\chi}_{[0,1)}$ decays like $2/\omega$ away from zero. Again we do $\mathbb{R}^+$ first.

$$\begin{aligned}
\int_0^\infty |\hat{g}(\omega)|^2 \, d\omega &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} |\pi(\omega)\hat{\chi}_{[0,1)}(\omega)|^2 \, d\omega \\
&\leq \sum_{n\in\mathbb{Z}} \sup_{[2^n,2^{n+1}]} |\hat{\chi}_{[0,1)}(\omega)|^2 2^n \int_1^2 |\pi(\omega)|^2 \, d\omega \\
&= \sum_{n\leq 0} 2^n \int_1^2 |\pi(\omega)|^2 \, d\omega + \sum_{n>0} \frac{4}{2^{2n}} 2^n \int_1^2 |\pi(\omega)|^2 \, d\omega \\
&= 6 \left\| \pi|_{[1,2)} \right\|_2^2 .
\end{aligned}$$

Completing the argument for $\mathbb{R}^-$ shows $\|\hat{g}\|_2 \leq \sqrt{6} \left\| \pi|_{\pm[1,2)} \right\|_2$. ■

In the folklore of dilation equations this could be considered surprising: many people think of the uniqueness result of [4]. This deals with $L^1(\mathbb{R})$ solutions where $\sum c_n = 2$. This can be applied to compactly supported $L^2(\mathbb{R})$ functions, which are all in $L^1(\mathbb{R})$.

Theorem 7 can clearly be used to calculate a basis for the $L^2(\mathbb{R})$ solutions to the equation $f(x) = f(2x) + f(2x-1)$, however we will not pursue this here. Theorem 7 can be extended to dilation equations which have a (non-zero $L^2(\mathbb{R})$) solution with analytic Fourier transform which decays like $|\omega|^{-p}$ for some $p > \frac{1}{2}$. This case includes the dilation equations used to build Daubechies' family of orthonormal wavelet bases.

Theorem 8. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which has the translation property for integers (1.2) and the dilation property for $2^{\mathbb{Z}}$ (1.3). Then $\mathcal{A}$ is of the form:*

$$\mathcal{A} = \pi\mathcal{F}$$

where $\pi(\omega) = \pi(2\omega)$ and $\pi \in L^\infty(\mathbb{R})$. Conversely any such π gives rise to a bounded linear $\mathcal{A}$ which has the dilation property for $2^{\mathbb{Z}}$ and the translation property for all reals.

Proof. Let $p = p_2$ be the function corresponding to $f(x) = f(2x)+f(2x-1)$. We know that $\mathcal{A}\chi_{[0,1)} \in F(p)$ and $\mathcal{F}\chi_{[0,1)}$ is maximal in $F(p)$. So $\mathcal{A}\chi_{[0,1)} = \pi\mathcal{F}g$, where $\pi(\omega) = \pi(2\omega)$ and $\pi \in L^2(\pm[1,2))$.

We note that $\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})}$ can be obtained by integer translations and dilations of scale 2^n applied to $\chi_{[0,1)}$. This allows us to make the following calculation:

$$\begin{aligned} \mathcal{A}\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})} &= \mathcal{A}\mathcal{D}_{2^m}\mathcal{T}_{-n}\chi_{[0,1)} &&= 2^{-m}\mathcal{D}_{2^{-m}}e^{in\cdot}\mathcal{A}\chi_{[0,1)} \\ &= 2^{-m}\mathcal{D}_{2^{-m}}e^{in\cdot}\pi\mathcal{F}\chi_{[0,1)} &&= \pi 2^{-m}\mathcal{D}_{2^{-m}}e^{in\cdot}\mathcal{F}\chi_{[0,1)} \\ &= \pi\mathcal{F}\mathcal{D}_{2^m}\mathcal{T}_{-n}\chi_{[0,1)} &&= \pi\mathcal{F}\chi_{[\frac{n}{2^m},\frac{n+1}{2^m})} \end{aligned}$$

Thus, as both $\mathcal{A}$ and $\pi\mathcal{F}$ are linear, we can see that $\mathcal{A}f = \pi\mathcal{F}f$ for any f in D. But this is a dense subset and $\mathcal{A}$ is continuous so if $\pi\mathcal{F}$ is continuous they agree everywhere. It is clear that if $\pi \in L^\infty(\mathbb{R})$ then $\pi\mathcal{F}$ will be continuous.

So it remains to show that $\pi \in L^\infty(\mathbb{R})$. If it were not we could consider π as an unbounded multiplier on a dense subset of $L^2(\mathbb{R})$, which would make $\mathcal{A}$ unbounded.

The converse is a simple matter of algebra and using $\pi(\omega) = \pi(2\omega)$. ■

Corollary 9. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which has the translation property for integers and the dilation property for $2^{\mathbb{Z}}$. Suppose also that $\mathcal{A}$ preserves inner products. Then $\mathcal{A}$ is of the form:*

$$\mathcal{A} = \pi\mathcal{F}$$

where $\pi(\omega) = \pi(2\omega)$ and $|\pi(\omega)| = \frac{1}{\sqrt{2\pi}}$ almost everywhere.

5 Shift and Dilation invariant Operators

By applying the inverse Fourier transform these results can be nicely summed up. The "translation property" now becomes "commutes with translation" and the "dilation property" becomes "commutes with dilation". We restate the results in this language.

Theorem 10. *Suppose $\mathcal{A} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator which commutes with translation by integers, then:*

1. *$\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ and $\mathcal{A}(\chi_{[0,1)}) = \chi_{[0,1)}$ implies $\mathcal{A} = \mathcal{I}$,*

2. *$\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ and $\mathcal{A}(\chi_{[0,1)}) \in L^1(\mathbb{R})$ implies $\mathcal{A} = c\mathcal{I}$,*

3. *$\mathcal{A}\mathcal{D}_n = \mathcal{D}_n\mathcal{A}$ for $n \in S \subset \mathbb{Z} \setminus \{0\}$ and $S^\times$ (the multiplicative group generated by S) is dense in $\mathbb{R}$ implies $\mathcal{A} = c\mathcal{I}$,*

4. *$\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ implies $\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$ where $\pi \in L^\infty(\mathbb{R})$ and $\pi = \mathcal{D}_2\pi$,*

5. *$\mathcal{A}\mathcal{D}_2 = \mathcal{D}_2\mathcal{A}$ and $\mathcal{A}$ is unitary implies $\mathcal{A} = \mathcal{F}^{-1}\pi\mathcal{F}$ where $|\pi(\omega)| = 1$ and $\pi = \mathcal{D}_2\pi$.*

Part 5, which is a reformulation of Corollary 9, has been proved in different ways in other contexts [2, 9].

This theorem has been stated with scale two in mind, but any scale $\lambda = 2, 3, 4, 5, \ldots$ would produce similar results.

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