Area, Proportion and Similarity in Euclid's Geometry

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Theories of Proportion

- **Numerical proportion** in elementary number theory (Euclid, *Elements*, Book VII. (Pythagorean.)
- **Equimultiple-based theory of proportion**, often attributed to Eudoxus of Cnidus. (Euclid, *Elements*, Book V.)
- **Proportion through ratios of real numbers**, such real numbers expressing ratios of magnitudes of the same species (Typical of elementary geometry from the mid nineteenth-century onwards.
- **Anthyphairetical theory of proportion**, an invention of the twentieth century related to the Euclidean Algorithm and continued fraction expansions.
- **An area-based theory of proportion?**

The 13 Books of Euclid's Elements of Geometry

1. Basic constructions;theories of triangles and parallelograms, theory of parallels, theory of area;
2. theory of squares and rectangles, 'geometrical algebra';
3. the theory of the circle;
4. the theory of circles inscribed in and circumscribed around triangles, and of regular polygons with 4, 5, 6 and 15 sides, with their inscribed and circumscribed circles;
5. **The theory of proportion attributed to Eudoxus;**
6. The theory of similarity for rectilineal plane figures (with a final proposition concerning circles);
7. elementary number theory and arithmetical proportionality;
8. more elementary number theory;
9. yet more elementary number theory;

10. a systematic investigation of certain types of straight line segment, incommensurable with a given line segment, that are classified as *incommensurable in square only*, *medial*, *binomial*, *apotome*, '*sides*' of binomial segments, '*sides*' of apotome segments;

11. introduction to stereometry, including discussion of volumes of parallelepipeds and prisms;

12. the theory of exhaustion, applied to investigations of the areas of circles, and of the volumes of tetrahedra and cones;

13. pentagon geometry, supplementing that developed in Book IV, together with constructions (with appropriate justification) of the five Platonic solids.

Ancient authors attributed the development of the material in Books X and XIII to Theaetetus of Athens (building on work of the Pythagoreans), and that in Books V and XII to Eudoxus of Cnidus.

Book V of Euclid's *Elements of Geometry* develops a theory of proportion, attributed by some to Eudoxus, based on the use of equimultiples of 'magnitudes'. We consider relationships amongst magnitudes of a given species (or kind). We supposed that such magnitudes can be summed together; appropriate rules apply. Also a lesser magnitude may be subtracted from a greater magnitude. In particular, given a magnitude α of an appropriate kind, and given also a natural number *n*, one can form the multiple $n\,.\,\alpha$ be adding together *n* copies of the magnitude α. Given magnitudes α, β, η and θ of the same kind, we say that η and θ are *equimultiples* of α and β if and only if there exists some natural number *n* for which $\eta = n\,.\,\alpha$ and $\theta = n\,.\,\beta$. Let us say that a species of magnitudes of the type described is *Archimedean* if, whenever a lesser magnitude is subtracted from a greater, multiples both of the part subtracted and also of the remainder exceed the whole. The thery of proportion developed in Book V of Euclid's *Elements of Geometry* is only applicable to such Archimedean species.

Given magnitudes α, β, γ, δ belonging to some Archimedean species, we say that α, β, γ, δ are *proportional* (by equimultiples), and write $\alpha : \beta \stackrel{E}{=} \gamma : \delta$, in those situations, and only those situations, where, for all natural numbers m and n,

$$\left\{ \begin{array}{l} \text{if } n \,.\, \alpha > m \,.\, \beta \text{ then } n \,.\, \gamma > m \,.\, \delta; \\ \text{if } n \,.\, \alpha = m \,.\, \beta \text{ then } n \,.\, \gamma = m \,.\, \delta; \\ \text{if } n \,.\, \alpha < m \,.\, \beta \text{ then } n \,.\, \gamma < m \,.\, \delta. \end{array} \right\}$$

Such conditions may be expressed more concisely by writing that, for all natural numbers m and n,

$$n \,.\, \alpha \begin{array}{c} > \\ = \\ < \end{array} m \,.\, \beta \textit{ entails } n \,.\, \gamma \begin{array}{c} > \\ = \\ < \end{array} m \,.\, \delta.$$

It is necessary to prove that certain rules concerning proportions are consequences of this definition.
This, in summary, is the theory of proportion attributed to Eudoxus of Cnidus, and developed in Book V of the *Elements*.

In his book *Le Géométrie Grecque*, published in 1887, Paul Tannery argued that "the discovery of incommensurabilty must have caused a veritable logical scandal in geometry and, in order to avoid it, they were obliged to restrict as far as possible the use of the principle of similitude, pending the discovery of a means of establishing it on the basis of a theory of proportion independent of commensurablity." (see Heath, *The Thirteen Books of Euclid's Elements*, 2nd edition, Vol. 2, p. 112, where the translation is given). This supposed 'foundations crisis' was further discussed by Hasse and Scholz in 1928. Those presuming the existence of this crisis would presumably have assumed, as Heath does, that the theory of proportion presented in Book V of the *Elements of Geometry* represented the resolution of the supposed foundations crisis.

In a paper published in 1933, Oskar Becker suggested that, before the theory of proportion involving comparison of equimultiples of magnitudes had been developed, Greek geometers had developed a theory of proportion based on *anthyphairesis*, incorporating methods related to the Euclidean algorithm that, for ratios of commensurable magnitudes, terminates thereby determining the greatest common measure of those magnitudes. Where the magnitudes constituting the ratio are incommensurable, the algorithm does not terminate (*Elements*, X.2), but infinite sequences of natural numbers arising from application of the algorithm would characterize the ratio. Representing the incommensurable ratio by an irrational number, the process corresponds to finding the continued fraction representation of that number.

Those who have investigated this approach include Wilbur Knorr and David H. Fowler.

Towards an area-based theory of proportion.

Notation:

Given straight line segments, we let $\mathrm{Rect}(K, L)$ represent, with regard to area, a rectangle with containing sides of length K and L that make a right angle with one another. We also let $\mathrm{Quad}(K)$ represent, with regard to area, a square whose sides are equal in length to K. Accordingly $\mathrm{Rect}(K, K) = \mathrm{Quad}(K)$.

We seek to show that a viable theory of *rectilineal proportion* can be formulated, where straight line segments K, L, M and N are said to be *rectilineally proportional* if and only if a rectangle with containing sides equal in length to K and N is equal in area to a rectangle with containing sides equal to L and M.
In symbols

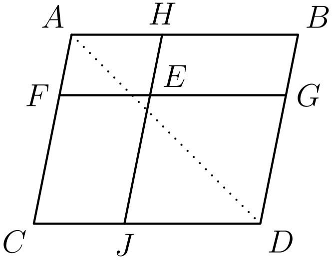
$$K : L \stackrel{R}{=} M : N \iff \mathrm{Rect}(K, N) = \mathrm{Rect}(L, M).$$

Many standard proporties of proportionality are easily verified when this definition is employed. However some work is necessary to show that, for straight line segments K, L, M, N, P, Q,

$$K : L \stackrel{R}{=} P : Q \text{ and } M : N \stackrel{R}{=} P : Q \Rightarrow K : L \stackrel{R}{=} M : N.$$

This encodes the principle that, for ratios of straight line segments, *ratios which are the same with the same ratio are also the same with one another* (see *Elements*, V.11).

Some Propositions



Proposition A. (*Elements*, I.43, with its obvious converse.) *Let ABDC be a parallelogram, let a point E be taken in the interior of this parallelogram, and let straight line segments FG and HJ passing through E be determined so that FG $\parallel$ AB and HJ $\parallel$ AC. Then the complements HBGE and FEJC are equal in area if and only if the point E lies on the diagonal AD.*

In what follows, given straight line segments L, M, $\mathrm{Rect}(L, M)$ represents, with respect to area, a *rectangle contained by* L, M. Such a rectangle has perpendicular containing sides equal to L and M.

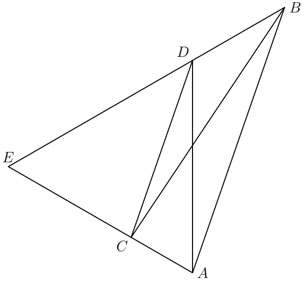
Proposition B. (Direct from *Elements*, I.44.) *Let K, L and M, be straight line segments. Then there exists a straight line segment N with the property that $\mathrm{Rect}(K, N) = \mathrm{Rect}(L.M)$.*

Proposition C. *Let K, L, M, N, P and Q be straight line segments. Suppose that $\mathrm{Rect}(K, Q) = \mathrm{Rect}(L, P)$ and $\mathrm{Rect}(M, Q) = \mathrm{Rect}(N, P)$, Then $\mathrm{Rect}(K, N) = \mathrm{Rect}(L, M)$.*

Proposition C, when established, would ensure that, with K, L, M, N, P and Q as in the statement of the proposition,

$$K : L \stackrel{R}{=} P : Q \text{ and } M : N \stackrel{R}{=} P : Q \Rightarrow K : L \stackrel{R}{=} M : N.$$

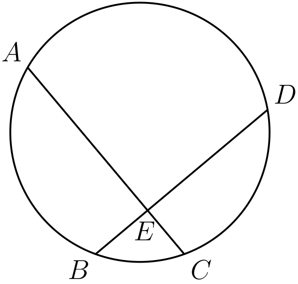
Proposition D. *Let EAB be a triangle, let points C and D be taken on the sides EA and EB respectively, and let C and D be joined. Then the lines AB and CD are parallel if and only if* $\mathrm{Rect}(EA, ED) = \mathrm{Rect}(EB, EC)$.

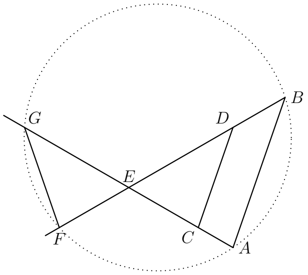


Proposition D, when established, would ensure that the sides AB and CD are parallel (and in consequence the triangles EAB and ECD are equiangular) if and only if $EA : EC \stackrel{R}{=} EB : ED$.

Proposition Z (Direct on applying *Elements*, III.35 and its obvious converse.) Given straight line segments AC and BD intersecting at E, the points A, B, C and D, lie on a circle if and only if

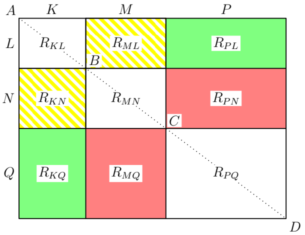
$$\mathrm{Rect}(AE, EC) = \mathrm{Rect}(DE, EB).$$





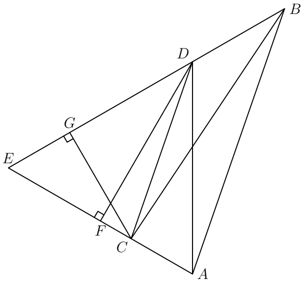
Proof of Proposition D using Circle Theorems. In the accompanying diagram $EF = EC$ and $EG = ED$. It follows that $AB||CD \iff \angle BAG = \angle DCE = \angle BFG$. This is the case if and only if A, B, G and F lie on a circle (by *Elements*, III.21 and its obvious converse), and thus if and only if $\mathrm{Rect}(EA, ED) = \mathrm{Rect}(EA, EG) = \mathrm{Rect}(EB, EF) = \mathrm{Rect}(EB, EC)$ (applying Proposition Z). ∎

Proposition C. *Let K, L, M, N, P and Q be straight line segments. Suppose that* $\mathrm{Rect}(K, Q) = \mathrm{Rect}(L, P)$ *and* $\mathrm{Rect}(M, Q) = \mathrm{Rect}(N, P)$, *Then* $\mathrm{Rect}(K, N) = \mathrm{Rect}(L, M)$.

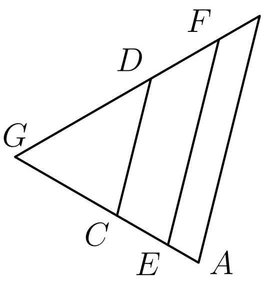


Proof of Proposition C using Application of Areas. $\mathrm{Rect}(M, Q) = \mathrm{Rect}(N, P) \Rightarrow B$, C, D are collinear. $\mathrm{Rect}(K, Q) + \mathrm{Rect}(M, Q) = \mathrm{Rect}(L, P) + \mathrm{Rect}(N, P) \Rightarrow A$, C, D are collinear. Consequently A, B, C are collinear, and hence $\mathrm{Rect}(K, N) = \mathrm{Rect}(L, M)$. ∎

Proposition D. *Let EAB be a triangle, let points C and D be taken on the sides EA and EB respectively, and let C and D be joined. Then the lines AB and CD are parallel if and only if* $\mathrm{Rect}(EA, ED) = \mathrm{Rect}(EB, EC)$.



Proof of Proposition D assuming Proposition C. $AB || CD \iff \triangle ACB = \triangle ADB \iff \triangle ECB = \triangle EAD \iff \mathrm{Rect}(EA, DF) = \mathrm{Rect}(EB, CG)$. But $\mathrm{Rect}(EC, DF) = \mathrm{Rect}(ED, CG)$. Consequently (applying Proposition C) $AB || CD$ if and only if $\mathrm{Rect}(EA, ED) = \mathrm{Rect}(EB, EC)$, as required. ∎



Proof of Proposition C, assuming Proposition D. Let K, L, M, N, P and Q be equal to GA, GB, GC, GD, GE and GF respectively. Now $\mathrm{Rect}(K, Q) = \mathrm{Rect}(L, P)$ and $\mathrm{Rect}(M, Q) = \mathrm{Rect}(N, P)$ imply that $AB \parallel EF$ and $CD \parallel EF$ (applying Proposition D). Then $AB \parallel CD$ (*Elements*, I.30), and hence $\mathrm{Rect}(K, N) = \mathrm{Rect}(L, M)$. ∎

Definition. Let K, L, M and N be straight line segments. We say that K, L, M and N are *rectilineally proportional*, and write $K : L \stackrel{R}{=} M : N$, if the rectangle contained by K, N, is equal in area to that contained by L, M, so that

$$\mathrm{Rect}(K, N) = \mathrm{Rect}(L, M).$$

Proposition F. *Let K, L, M, N, P and Q be straight line segments. Suppose that*

$$K : L \stackrel{R}{=} P : Q \text{ and } M : N \stackrel{R}{=} P : Q.$$

Then $K : L \stackrel{R}{=} M : N$.

Proposition G. *Let K, L, M, P, Q and R be straight line segments satisfying the conditions.*

$$K : P \stackrel{R}{=} L : Q \text{ and } P : M \stackrel{R}{=} Q : N.$$

Then, ex aequali, $K : M \stackrel{R}{=} L : N$.

Now let X and Y be plane rectilineal figures, and let P and Q be straight line segments. The rectlineal figures can be applied to the line Q to produce straight line segments K and L that are such as to ensure that X is equal in area to the rectangle contained by Q, K and Y is equal to the rectangle contained by Q, L. (This follows on applying *Elements*, I.45.) Accordingly

$$X = \mathrm{Rect}(K, Q) \text{ and } Y = \mathrm{Rect}(L, Q).$$

Similarly there exist straight line segments M and N for which

$$X = \mathrm{Rect}(M, P) \text{ and } Y = \mathrm{Rect}(N, P).$$

Then $\mathrm{Rect}(K, Q) = \mathrm{Rect}(M, P)$ and consequently $K : M \overset{R}{=} P : Q$. Similarly $L : N \overset{R}{=} P : Q$. It follows from Proposition F that $K : M \overset{R}{=} L : N$, and accordingly, by alternation, $K : L \overset{R}{=} M : N$.

Accordingly (in view of the transitivity result proved in Proposition F), it makes sense to define proportions involving both straight line segments and rectilineal figures so that straight lines segments R and S and rectilineal figures X and Y satisfy the proportion $X : Y \overset{R}{=} R : S$ if and only if line segments Q, K and L can be determined so that $X = \mathrm{Rect}(K, Q)$, $Y = \mathrm{Rect}(L, Q)$ and $K : L \overset{R}{=} R : S$.

Similarly, given rectilineal plane figures X, Y, Z and W, we can say that the rectilineal plane figures satisfy the proportion $X : Y \overset{R}{=} Z : W$ if and only if straight line segments Q, K, L, M and N can be determined so as to ensure that

$$X = \mathrm{Rect}(K, Q), \quad Y = \mathrm{Rect}(L, Q),$$

$$Z = \mathrm{Rect}(M, Q), \quad W = \mathrm{Rect}(N, Q)$$

and $K : L \overset{R}{=} M : N$.

"We must not fail to observe that we often fall into error because our conclusion is not in fact primary and commensurately universal in the sense in which we think we prove it so. We make this mistake (1) when the subject is an individual or individuals above which there is no universal to be found: (2) when the subjects belong to different species and there is a higher universal, but it has no name: [. . .] An instance of (2) would be the law that proportionals alternate. Alternation used to be demonstrated separately of numbers, lines, solids and durations, though it could have been proved of them all by a single demonstration. Because there was no single name to denote that in which numbers, lengths, durations and solids are identical, and because they differed specifically from one another, this property was proved of them separately. To-day, however, the proof is commensurately universal, for they do not possess this attribute qua lines or qua numbers, but qua manifesting this generic character which they are postulated as possessing universally."

Aristotle, *Posterior Analytics*, I.5, 74 a 4–25.