The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 41

Transcribed by David R. Wilkins

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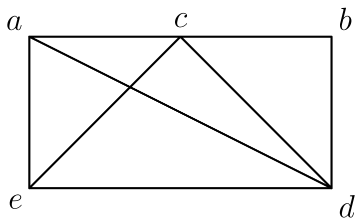
[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 191–194 (1792).]

PROPOSITION XLI. Theorem XXXI.

If a parallelogram has the same base with a triangle, and is between the same parallels, the parallelogram shall be double the triangle.

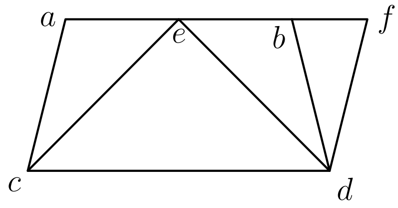
The present theorem also is local, but it mingles the constructions of triangles and parallelograms, situated under the same altitude. As therefore, we have separately surveyed parallelograms and triangles, so when we assume each of them in conjunction, and with the same condition, we contemplate their proportion to each other. In the former, therefore, an equality of proportion is apparent, since all upon the same bases, and between the same parallels, have a mutual equality, whether they are triangles, or parallelograms. But in these latter, the first of unequal proportions, I mean the duple, is exhibited: for he demonstrates that a parallelogram is double of a triangle, on the same base, and possessing the same altitude. But the elementary institutor shews the thing proposed, by supposing the vertex of the triangle external to the parallelogram. We can, however, demonstrate the consequence, by assuming the line which is parallel to their common base, in the other side of the parallelogram: for these are two cases of the theorem. Since in consequence of the two having the same base, it is necessary that the vertex of the triangle should either be within, or without the parallelogram. Let there be, therefore, a parallelogram $a\,b\,e\,d$, and a triangle $e\,c\,d$, and let a point c be placed between the points a and b, and connect the right line $a\,d$. Because, therefore, the parallelogram is double of the triangle $e\,c\,d$, but the

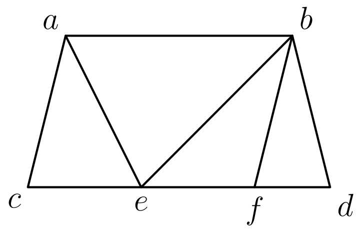


triangle $a\,d\,c$, is equal to the triangle $e\,d\,c$, hence, the parallelogram is double of the triangle $e\,c\,d$. And hence it is evident that a parallelogram is double of a triangle on the same base. But if the bases are equal, we can shew the same by drawing the diameters of the parallelograms: for if the triangles are equal, the parallelogram which is double of the one, will also be double of the other. But triangles are equal, on account of the equality of bases, and the identity of altitude. The geometrician, therefore, very properly omits this, for the demonstration is the same: since they will either have the same part, or they will be conjoined in one point only, or they will be separate from each other. But in whatever manner they may receive this variety, there is one demonstration according to all the cases.

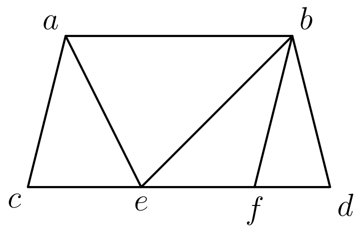
We can likewise demonstrate the converse propositions to this theorem, after the same manner. One of which is: *If a parallelogram is double of a triangle, and they have the same or equal bases, and are at the same parts, they shall be between the same parallels.* For if they are not the whole shall be equal to the part, and the same proportion shall prevail: since it is necessary that the vertex of the triangle should either fall within, or external to the parallels. But in either case, the same impossibility will be the result, but drawing a parallel to the base, through the vertex of the triangle. But the second converse theorem is: *If a parallelogram is double of a triangle, and both are between the same parallels, they will either be situated upon one base, or upon equal bases.* For if they are upon unequal bases, since we have assumed the figures to be equal, we may shew that the whole will be equal to its part. Hence, all these theorems end in this common impossible: and on this account, the institutor of the Elements leaves us to investigate the variety they contain, as he himself, has contracted his speculation to such as are more simple, and of a more primary nature. However, as we have recognized these observations, let us see for the sake of exercise, by not assuming a parallelogram, but a trapezium, two of whose sides only are parallel (because it has the same base with the triangle, while it is situated between the same parallels), let us, I say, consider what proportion it possesses to the triangle. That it has not, therefore, a duple proportion is evident: for if it had a duple ratio, it would be a parallelogram, since it is a quadrilateral figure. But I say that it is either greater than double or less for since the two sides are parallel, one is greater, but the other less; because if equal, the sides conjoining them will be parallel. If, therefore, the triangle has the greater side for the base, the quadrilateral figure will be less than double of the triangle: but if the lesser side, it will be more than double. For let $a\,b\,c\,d$, be a quadrilateral figure, and let the side $a\,b$, be less than the side $c\,d$, and produce the side $a\,b$, in infinitum, and let the triangle $e\,c\,d$ have the same base with the quadrilateral figure, that is $c\,d$; and lastly, through d, draw $d\,f$, parallel to $a\,c$. Hence, the parallelogram $a\,c\,d\,f$, is double of the triangle $e\,c\,d$; and so the quadrilateral figure $a\,b\,c\,d$, is less than the double of the triangle.

Again, let the triangle have the base $a\,b$, and draw $b\,f$, parallel to $a\,c$. The parallelogram, therefore, $a\,b\;\;f\,c$, is double of the triangle. And hence, the





quadrilateral figure, $abcd$, is more than double of the triangle. This being shewn; we affirm, that when there is a quadrilateral figure, whose two opposite sides only, are parallel, if one of the parallel lines is bisected, right lines are drawn from it to the other side, the quadrilateral figure, is either more or less than double of the triangle resulting from such a construction. But if one of the sides by which the parallel lines are conjoined, is bisected, and certain right lines are drawn from it to the remaining side, the quadrilateral figure, will be perfectly double of the triangle which is produced. And this may be shewn as follows. Let there be a quadrilateral figure $abcd$, and let the side ad, be paralel to the side bc, and bisect dc, in the point e, and connect the right lines ae, eb, and produce be, till it coincides with ad, in some point, as f. Because, therefore, the angles at the point e, are equal, for they are



vertical; likewise, because the angle fde, is equal to the point bce, the side also fe, will be equal to the side eb, and the triangle def, will be equal to the triangle bce. Let the common triangle ade, be added. The whole triangle, therefore, aef, is equal to the two triangles ade, bce. But the triangle aef, is equal to the triangle aeb: for they are upon equal bases be, ef, and between the same parallels, if a line parallel to bf, is drawn[189]. Hence,

[189]Barocius is of opinion, that this Commentary was originally mutilated; and that the

the triangle $a\,e\,b$, is equal to the triangles $a\,d\,e$, $b\,c\,e$, and the quadrilateral figure $a\,b\,c\,d$, is double of the triangle $a\,e\,b$, which was to be shewn. After the same manner, we may shew, that if the side $a\,b$ is bisected, and certain right lines are drawn from it, to the side $e\,d$, the quadrilateral figure will be double of the triangle formed by such a construction. If, therefore, one of the sides by which the parallel lines are conjoined is bisected, and from it certain right lines are drawn to the remaining side, the quadrilateral figure shall be double of the triangle. And these things are demonstrated for the sake of geometrical exercise. Let us now proceed to the subsequent propositions.

PROPOSITION XLII. Problem XI.

> To construct a parallelogram equal to a given triangle, in a given rectilineal angle[190].

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part which follows the word *drawn*, was added by some skilful geometrician, as necessary to the perfection of the demonstration. See his Scholium to this Commentary.

[190]The Commentary of Proclus, on this proposition is wanting in the Greek, and, as we are informed by Barocius, in all the MS. copies which he had an opportunity of consulting. Barocius has endeavoured to supply this deficiency, after the manner of Proclus; but he appears to have fallen into prolixity, by a too minute division of the problem