The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 35

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[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 177–183 (1792).]

PROPOSITION XXXV. Theorem XXV.

Parallelograms which are upon the same base, and between the same parallels, are equal to each other.

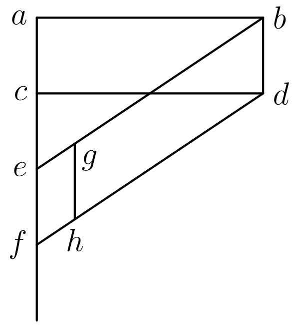
As we have said that of theorems, some are universal, but others particular, and as dividing these we have subjoined, that some are also simple, but others composite, and have shewn the nature of each, so according to another distinction, we assert that some of these are local, but others non-local. But I call those local, to which the same symptom happens in a certain place; and I denominate the place of the line or a superficies, that situation, which produces one and the same symptom. For of local theorems some are constructed in lines, but others in superficies. And because of lines, some are plane, but others solid, the plane being those of which there is a simple conception in a plane, as of a right line: but the solid those whose origin appears from a certain section of a solid figure, as of cylindric, spiric, and conic lines, I should say, that of the local theorems which are constructed in lines, some have a plane, but others a solid place. The present theorem, therefore, is both local, and local in lines, and a plane. For the whole space which lies between the parallels, is the place of the parallelograms constructed upon the same base; and which the institutor of the Elements shews to be equal to each other. But of those local theorems which are called solid, let the following be an example[181]. *The parallelograms which are inscribed within the asymptotes and the hyperbola are equal*: for it is evident that the hyperbola is a solid line.

But Chrysippus, as we are informed by Geminus, assimilates theorems of this kind to ideas. For as ideas comprehend the origin of infinites in terminated limits, so in these also there is a comprehension of infinites, in terminated places, and by the boundary equality appears, since the altitude of the parallels remaining the same, if infinite parallelograms are conceived upon the same base, they may all be shewn to be equal to each other. The present,

[181]This is a well known property of the hyperbola, and its asymptotes; and is thus expressed by Mr. Simpson, in his Conic Sections, Lib. 3. Prop. 16. "If from a point in the hyperbola, any two lines are drawn to the asymptotes, and if from any point, in the same or opposite hyperbolas, there are drawn to the asymptotes other right lines parallel to the former; then the rectangle contained by the lines first drawn, shall be equal to the rectangle contained by the other drawn lines."

therefore, is with the institutor of the Elements, the first local theorem. And he appears, when, agreeable to an elementary mode, he had distinguished theorems by a variety, according to all possible divisions, with great propriety not to have omitted, considering their idea of this kind. Nevertheless, as his discourse, for the present, is concerning right lines, he delivers local plane theorems in right lines: but in the third book, as he treats concerning things which may be contemplated of circles, and their symptoms, he likewise teaches the particulars, which are constructed in circumferences belonging to local, and at the same time, plane theorems. And such, among these, is the theorem, which says, *that angles in the same segment, are equal to one another.* Also this which asserts, *that the angles in a semicircle are right.* For if infinite angles are constructed in a circumference, the same base remaining, they are all shewn to be equal: but if that which is comprehended by the base and the circumference, is a semicircle, they are all shewn to be right. And these, indeed, correspond in proportion to triangles and parallelograms upon the same base, and between the same parallels. And such is the species of theorems called local, by the ancient mathematicians.

But perhaps it may seem worthy of admiration, to such as are unskilled in contemplations of this kind, that parallelograms constructed upon the same base, and between the same parallels, should be equal to each other. For it may be asked, how is this possible, since the longitude of the spaces, constructed on the same base, increases in infinitum? Since as much as we produce the parallels, by so much we may also increase the longitudes of the parallelograms. But some one may not improperly enquire how, while this takes place, the equality of the spaces remains. For if the breadth is the same (since the base is one), but the length is greater, will not the space also be greater? The present theorem, therefore, and that which follows concerning triangles, are among the number of mathematical theorems, which are denominated admirable. For mathematicians in theorems, as the Stoics in arguments, have established a *place*, which is called admirable, and they place the present among theorems of this kind. The vulgar, therefore, are immediately astonished, when they hear that the multiplication of length does not destroy the equality of spaces on the same base. We must nevertheless assert, that equality and inequality possess the greatest power of increasing or diminishing the spaces of angles. For in proportion as we make angles unequal, in such proportion we diminish the space, if the length and breadth remain the same. Hence the increase of length is necessary, that we may preserve equality. Thus for example, let there be a parallelogram $a\,b\,c\,d$, and let the side $a\,c$ be produced in infinitum, and let it be a right-angled parallelogram; and lastly, on the base $b\,d$, construct another parallelogram $b\,e\,f\,d$. That the length, therefore, is increased is evident: for the side $b\,e$, is greater



than the side $a\,b$, since the angle at the point a is right. But this necessarily takes place, as the angles of the parallelogram $b\,e\,f\,d$, are unequal, and some of them are acute, but others obtuse: and this happens, because the side $b\,e$, approaches after a manner to the side $b\,d$, and contracts the space. For let $b\,g$ be taken equal to $a\,b$, and through g, draw $g\,h$ parallel to $b\,d$. The length, therefore, of the parallelogram $b\,d\,g\,h$, is equal to the length of the parallelogram $a\,b\,c\,d$, and the breadth is the same, and yet one space is less than the other; for it is less than $b\,e\,f\,d$. Hence, the inequality of angles diminishes the area, but the increment of length adding as much as the inequality of angles takes away, preserves the equality of the spaces. But the boundary of the increase of length, is the place of the parallel lines. For when both parallelograms are rectangular, and have an equal ambit, the quadrangle is shewn to be greater than the oblong[182]: but when they are both equilateral, and consequently have an equal ambit, that which is rectangular, is shewn to be greater than this which is non-rectangular[183]. For the rectitude of angles, and the equality of sides, possesses universal power in the augmentation of spaces. It is on this account that a quadrangle is the greatest of all figures with an equal ambit, and a rhomboides the least. And these observations we shall demonstrate in another place[184]: for they more properly belong to the hypotheses of the second book.

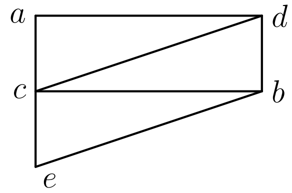
But with respect to the present theorem, it is requisite to know, that when Euclid calls parallelograms equal, he means the spaces, and not the sides:

[182]Thus let there be a square, whose side is equal to three, and a parallelogram whose longest side is equal to four, and its shortest to two; the ambit of each figure, will indeed be equal to twelve, but the area of the square will be equal to nine, and of the parallelogram to eight.

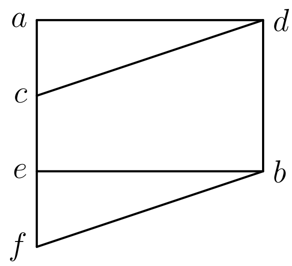
[183]This will be evident by conceiving a rectangular parallelogram equal to that which is non-rectangular, described on the same base: for the ambit of the former will be less than that of the latter, and consequently less than the parallelogram, with an ambit equal to the non-rectangular parallelogram.

[184]From hence it is evident that it was the intention of Proclus to comment on the whole of Euclid: but it does not appear that he every carried this design into execution.

for he now discourses of areas. And we must likewise observe, that he first mentions trapeziums in the demonstration of this theorem: from whence also it is manifest, that he does not improperly teach us concerning a trapezium, in the definitions, when he informs us that it is indeed of a quadrilateral species, but is not a parallelogram. For the figure which has not its opposite sides and angles equal, falls from the order of the parallelograms. The institutor of the Elements, therefore, as he had chosen a more difficult case, demonstrates the thing proposed. But if any should say, let the parallelograms $a\,b\,c\,d$, and $b\,d\,c\,e$, be on the same base $d\,b$, so that the side $c\,d$ may be the diameter of the parallelogram $a\,b$, we can shew that according to this position they are equal. For the triangle $b\,c\,d$, is half of each parallelogram: because $c\,d$ is the



diameter of $a\,b$, but $c\,b$ of $d\,e$; and diameters bisect parallelograms. Hence $a\,b$ is equal to the parallelogram $d\,e$. Again, if any one should suppose that the side $a\,c$, of the parallelogram $a\,b$, is cut by the side $d\,c$, and that the parallelograms are situated as $a\,d\,b\,e$, $b\,d\,c\,f$, we can shew that these also are equal. For since the side $a\,e$, is equal to the side $c\,f$ (each because opposite being equal to $d\,b$), let the common right line $c\,e$ be taken away. Hence $a\,c$ is equal to $e\,f$. But $a\,d$, also, is equal to $e\,b$, and the angle $c\,a\,d$, to the angle $f\,e\,b$. For $a\,d$ is parallel to $e\,b$; and hence, the base $c\,d$, is equal to the base



$f\,b$, and the whole triangle $a\,d\,c$, is equal to the whole triangle $e\,b\,f$. Let the common trapezium $c\,b$, be added. The whole, therefore, $a\,b$, is not unequal to the whole $d\,f$. And here you may observe that these are the only three cases. For the side $d\,c$, either cuts the side $e\,b$, according to the position of the elementary institutor; or it falls on the point c, as in the penultimate description: or it cuts the line $a\,e$, according to the present supposition. And thus the theorem is shewn to be true according to all its cases. Lastly, as there is a two-fold difference of trapeziums, and one kind has neither of the opposite sides parallel, but the other has one side parallel to one, this latter species of trapeziums is alone employed by the geometrician throughout the elements, and in the present description: for $c\,e$ is parallel to $d\,b$.