The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 32

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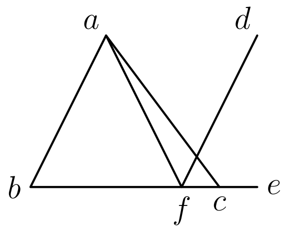
[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 162–169 (1792).]

PROPOSITION XXXII. Theorem XXII.

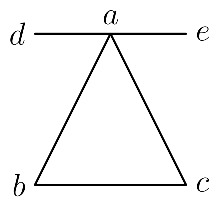
> One side of every triangle being produced, the external angle of the triangle is equal to the two internal and opposite angles; and the three internal angles of a triangle are equal to two right angles.

As much as was deficient in the sixteenth and seventeenth theorem, so much Euclid adds in the present. For we not only learn by this theorem that the external angle of a triangle is greater than either of the internal and opposite angles, but likewise how much it is greater; since as it is equal to both, it is greater than either of the remaining angles. Nor do we alone know from this theorem, that any two angles of a triangle, are less than two right, but by how much they are less: for they are deficient by the remaining third. The former, therefore, were more indefinite theorems: but this brings with it, on both sides, a boundary to science. We must not, however call them on this account superfluous: for they are of the greatest utility in many demontrations; and the present is proved by their assistance. And besides this, it is necessary that our knowledge, proceeding from the imperfect to the perfect, should pass from indeterminate apprehensions, to determinate and cetain propositions. But the institutor of the Elements, by drawing a parallel externally, exhibits each of the objects of the investigation. It is, however, possible that the same thing may be shewn without drawing the parallel externally; and this, by only changing the order of the things exhibited. For Euclid first shews, that the external angle is equal to the internal and opposite, and from this he proves the remainder. But we shall demonstrate this by a contrary mode of proceeding. Let there be then a triangle $a\,b\,c$, and let the side $b\,c$ be produced to the point e. Then take a point f in $b\,c$, and connect $a\,f$, and through the point f, let $f\,d$ be drawn parallel to $a\,b$. Because, therefore, $f\,d$ is parallel to $a\,b$, and a right line $a\,f$, falls upon these parallels, as also a right line $b\,c$, hence the alternate angles are equal, and the external is equal to the internal angle. The whole, therefore $a\,f\,c$, is equal to $f\,a\,b$, added to $a\,b\,f$. In like manner we may shew by drawing a parallel, that the angle $a\,f\,b$, is equal to the angles $f\,a\,c$, $a\,c\,f$. The two, therefore, $a\,f\,b$, $a\,f\,c$, are equal to the three angles of the triangle: and hence, the three angles of a triangle are equal to two right, viz. to $a\,f\,b$, added to $a\,f\,c$. But



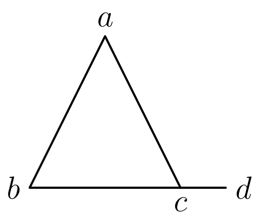
$a\,c\,f$, $a\,c\,e$, are also equal to two right angles. Let, therefore, the common angle $a\,c\,f$, be taken away; and then the remaining external angle will be equal to the internal and opposite angles. And after this manner may the present theorem be exhibited.

But Eudemus, the Peripatetic, ascribes the invention of this theorem to the Pythagoreans, I mean that every triangle has its internal angles equal to two right, and says that they demonstrate it in the following manner. Let there be a triangle $a\,b\,c$, and let there be drawn through the point a, a line $d\,e$, parallel to $b\,c$. Because, therefore, the right lines $d\,e$, $b\,c$, are parallel, the



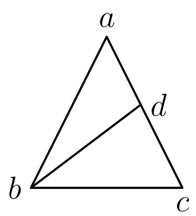
alternate angles are equal. Hence, the angle $d\,a\,b$, is equal to the angle $a\,b\,c$; and the angle $e\,a\,c$, to the angle $a\,c\,b$. Let the common angle $b\,a\,c$, be added. The angles, therefore, $d\,a\,b$, $b\,a\,c$, $c\,a\,e$, that is, the angles $d\,a\,b$, $b\,a\,e$, and that is two right, are equal to the three angles of the triangle. And such is the demonstration of the Pythagoreans.

But it is here requisite to deliver such theorems as are converse to the present theorem of the elementary institutor. For two are converted to one, since this is a composite, both, according to *the object of enquiry*, and the *datum*: for the *datum* is two-fold, viz. the triangle, and one of its sides produced; and in like manner *the object of enquiry.* For one part says, that the external angle is equal to the internal and opposite angles: but the other, that the three internal angles are equal to two right. If therefore, we suppose that the external is equal to the internal and opposite angles, we may shew that one side is produced, and that the right line externally situated, is in a direct position with one of the sides of the triangle: but if the three internal angles are equal to two right, we may shew that the given figure is a triangle. And so *the whole object of enquiry*, is converse to *the whole datum.* Let there be then a triangle $a\,b\,c$, and let the external angle $a\,c\,d$, be equal to the internal and opposite angles, I say that the side $b\,c$, is produced to the point d, and that $b\,c\,d$, is one right line. For since the angle $a\,c\,d$, is equal

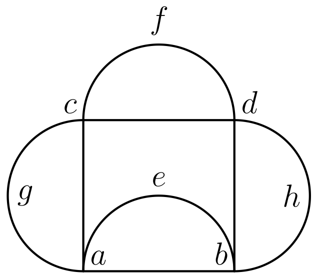


to the internal and opposite angles, let the common angle $a\,c\,b$ be added. The angles, therefore $a\,c\,d$, $a\,c\,b$, are equal to the three angles of the triangle $a\,b\,c$. But the three angles of the triangle $a\,b\,c$, are equal to two right. And hence, the angles $a\,c\,d$, $a\,c\,b$, are equal to two right. But if two right lines being consequently placed, an not at the same parts to any right line, and at a point in it, make the successive angles equal to two right, those right lines shall be in a direct position to each other. The right line, therefore $b\,c$, is in a direct position to $c\,d$.

Let there be again, a certain right lined figure $a\,b\,c$, having three angles alone equal to two right, viz. a, b, and c, I say that the figure is a triangle, and that $a\,c$, is one right line. For let the right line $b\,d$ be connected. Because,



therefore, the three angles of each of the triangles $a\,b\,d$, $b\,d\,c$, are equal to two right, of which the angles of the figure $a\,b\,c$, are equal to two right, the remainders $a\,d\,b$, $c\,d\,b$, are equal to two right, and they are placed about a right line $b\,d$. Hence $d\,c$, is in the same direction with $d\,a$; and so the side $a\,c$ is one right line. In like manner we may shew that the side $a\,b$, and the side $b\,c$, are each of them one right line. And consequently the figure $a\,b\,c$, is a triangle. If then a figure having internal angles equal to two right, is right lined, it is perfectly a triangle: but it does not follow that a figure is a triangle merely because it has internal angles equal to two right. For you will find a figure constructed from circumferences, having its internal angles equal to two right. For let there be a quadrangle $a\,b\,c\,d$, and upon the side $a\,b$, let a semicircle $a\,e\,b$, be internally described: but on upon the other sides, let the semicircles be externally described, as f, g, h. The figure, therefore,



which is comprehended by the semicircles, has two angles $g\,a\,e$, $e\,b\,h$, equal to two right, viz. to $c\,a\,b$, $d\,b\,a$. For this was shewn in the petitions[175], and these angles alone are in this figure. There is, therefore, a certain figure not a triangle, which has its internal angles equal to two right. And thus much may suffice concerning converse theorems.

But as we have discovered that the three angles of every triangle are equal to two right, we ought to determine a certain method, by which we may find how many angles, of all other multangles[176], are equal to so many right angles; as for instance, of a quadrangle, quinquangle, and of all conseqeunt multilateral figures. In the first place, therefore, it must be known, that every right-lined figure may be resolved into triangles, since a triangle is the principle of the constitution of all things, which Plato also asserts in the Timæus, when he teaches us that the rectitude of a plane basis is composed from triangles. But every figure is resolved into triangles less in number, by the binary, than its proper sides. If a quadrilateral figure, into two triangles: if a figure of five sides, into three: if of six sides, into four. For two triangles composed together, immediately form a quadrilateral figure. But the number of composite triangles by which the first constituted figure differs from its sides, is the measure of difference to the rest. Hence, every multilateral figure possesses more sides, by the binary, than the triangles into which it may be dissolved. But every triangle has been shewn to contain angles equal to two right. And hence, if the number of the angles be made double to that of the composite triangles, it will afford a multitude of right angles, to which the angles of every multangle will be equal. On this account every quadrilateral figure has angles equal to four right, since it is composed from two triangles: but every figure of five sides has angles equal to six right; and after the same manner of the rest in a consequent order. This one thing, therefore, is to be assumed from the present theorem, concerning all multangular and right-lined figures.

But there is another consequent to this, which is summarily as follows.

[175] In Lib. III. Com. 2.

[176] [DRW—*i.e.*, polygons]

In every right-lined figure, each of its side being at the same time produced, the angles externally constituted are equal to four right. For it is requisite, indeed, that the successive right angles should be double of the multitude of the sides; because, in each they are constituted equal to two right. But the right angles equal to the internal angles being taken away, the remaining external angles are equal to four right. As for example, if the figure is a triangle, while every one of its sides is produced, at the same time internal and external angles are constructed equal to six right angles, of which the internal angles are equal to two right, but the remaining external angles to four right. But if the figure be quadrilateral, the angles are in all eight, since they are double of the sides, of which the internal are equal to four right, and the external to the four remaining angles, and the consequences will be similar in infinitum. But after these observations we may also collect, that by this theorem every angle of an equilateral triangle is two thirds of a right angle: but that an isosceles triangle, when the vertical angle is right, has each of its remaining angles the half of one right, as a semiquadrangle; and that a scalene triangle, when it is half of an equilateral triangle, formed by a perpendicular drawn from any angle to its opposite side, has one angle right, but the other (which likewise belonged to the equilateral triangle) two thirds of a right angle, and the remainder by a necessary consequence, a third part of a right angle. For it is requisite that the three should be equal to two right. But I do not conceive that these remarks are foreign from our purpose, since they prepare us for the doctrine of Timæus. This also must be observed, that the possession of internal angles equal to two right, is inherent essentially, and answering to the predication *according to what*, in a triangle. And on this account, Aristotle in his Treatise on Demonstration[177], employs this as an example, considering it *according to what.* As therefore *to be terminated*, is essentially and primarily inherent in every figure, so likewise the possession of internal angles equal to two right, is essentially and primarily inherent in a triangle, though not in every figure. And the truth of this theorem seems to present itself to us according to common conceptions. For if we conceive a right line, and two right lines standing on its extremities, and inclining to each other, so as to form a triangle, we shall find that in proportion to their inclination they diminish the right angles, which they form with the right line. Hence, obtaining as much angular quantity, by their inclination at the vertex, as they take away from the base, they necessarily form three angles equal to two right.

[177]i.e., In his last analytics. See the second section of the Dissertation, Vol. I. of this work.