The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 29

Transcribed by David R. Wilkins

August 2020

[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 152–159 (1792).]

PROPOSITION XXIX. THEOREM XX.

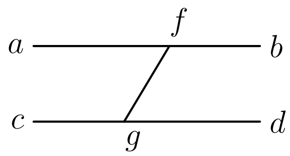
> A right line falling upon parallel right lines, makes the alternate angles equal to each other; and the external equal to the internal angle, oppositely situated, and at the same parts; and the internal angles at the same parts equal to two right.

The present theorem is converted in both the preceding. For that which is *the object of investigation*, in each of them, forms the hypothesis: but what are *data* in the preceding, he proposes to shew in the present. And this difference of converse theorems is not to be passed over in silence. I mean that every thing which is converted, is either converted as one to one, as the sixth proposition to the fifth; or as one to a many, as the present to the preceding; or as many to one, as will shortly be manifest[171]. But in the present theorem, the institutor of the Elements first employs the petition, which says: *If a right line falling upon two right lines, makes the angles situated internally, and at the same parts less than two right, those right lines whilst they are infinitely produced, will coincide at those parts in which the angles less than two right are situated.* But in our exposition of things prior to theorems [172], we have asserted, that this petition is not allowed by all to be indemonstrably evident. For how can this be the case when its converse is delivered among the theorems as demonstrable? For the theorem which says that the two internal angles of every triangle are less than two right, is the converse of this petition. Besides, the perpetual inclination of right lines, more and more, while they are produced, is not a certain sign of coincidence, because other lines are found, perpetually inclining, and never coinciding, as we have already observed. Formerly, therefore, some, when they had pre-ordained this as a theorem, considered that which is assumed by the institutor of the Elements as a petition, to be worthy of demonstration. But this seems to be shewn by Ptolemy himself, in a book entitled: *That right lines which are produced from less than two right angles, coincide.* And this he proves by pre-assuming many things, which as far as to the present theorem, are already demonstrated by the elementary institutor; and he supposes that all

[171]In the Comment on the 32d proposition.

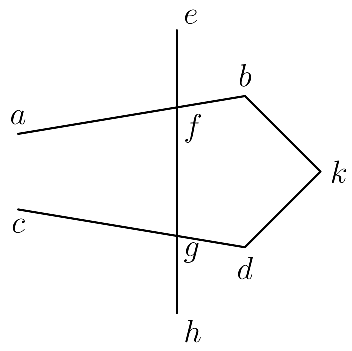
[172]In book III. chap. I. and in Com. 3.

are true (lest we should also superadd another confusion) and that this, as a small assumption, may be exhibited from the preceding. But this also is one of the things previously exhibited, which says, *that the right lines produced from two angles equal to two right, will never coincide.* I say, therefore, that the converse also is true, which says, *that right lines being parallel, if they are cut by one right line, the angles situated internally, and at the same parts, shall be equal to two right angles.* For it is necessary that a line cutting parallels, should either make the angles internally situated, and towards the same parts, equal to two right, or less, or greater than two right. Let the lines then, $a\,b$, $c\,d$, be parallel, and let the right line $g\,f$ fall upon them, I say that it will not make the angles internal, and at the same parts greater than two right. For if the angles $a\,f\,g$, $c\,g\,f$, are greater than two right, the remainders



$b\,f\,g$, $d\,g\,f$, are less than two right. But the same are also greater than two right. For $a\,f$, and $c\,g$, are not more parallel than $f\,b$, and $g\,d$. Hence, if the line which falls on $a\,f$, $c\,g$, makes the internal angles greater than two right, that also which falls upon $f\,b$, $g\,d$, will make the internal angles greater than two right. But they are also less than two right (since the four, $a\,f\,g$, $c\,g\,f$, $b\,f\,g$,, $d\,g\,f$, are equal to four right) which is impossible. In like manner we may plainly shew, that the right line which falls on parallels, does not make the angles internal, and at the same parts, less than two right. But if it makes them neither greater nor less than two right, it remains that the incident line must make the angles internal, and at the same parts equal to two right. This then being previously shewn, the thing proposed, is doubtless demonstrated. For I say, that if a right line falling upon two right lines, makes the angles situated internally, and at the same parts, less than two right, if those lines are produced they will concide at those parts in which the angles less than two right are situated. For let them not coincide. But if they are non-coincident at those parts in which the angles less than two right are situated, much more will they be non-coincident at the other parts, in which the angles greater than two right are situated. Hence the right lines will be non-coincident at both parts; and if this be true, they will be parallel. But it was shewn that the right line which falls on parallels, makes the angles internal and at the same parts equal to two right. The same therefore, are both equal to, and less than two right, which is impossible.

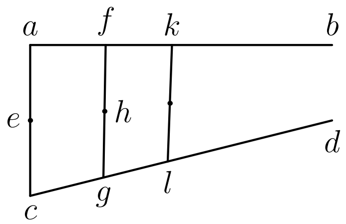
Ptolemy having previously shewn this, and proceeding to the thing proposed, wishes to add something more accurate, and to shew that if a right line falling upon two right lines, makes the angles internal, and at the same parts, less than two right, the lines are not only coincident as has been shewn, but likewise that their coincidence takes place at those parts, in which the angles less than two right, and not at those in which the angles greater than two right are situated. For let there be right lines $a\,b$, $c\,d$, and let a right line $e\,f\,g\,h$, falling upon them make the angles $a\,f\,g$, and $c\,g\,f$, less than two right. The remainders, therefore, are greater than two right; and thus it is



shewn that the right lines coincide. But if they coincide, they will either coincide at the points a and c, or at the points b and d. Let them coincide at the points b and d in the point k. Because, therefore, the angles $a\,f\,g$, and $c\,g\,f$, are less than two right, but the angles $a\,f\,g$, $b\,f\,g$, are equal to two right, by taking away the common angle $a\,f\,g$, the angle $c\,g\,f$, will be less than the angle $b\,f\,g$. The external angle, therefore, of the triangle $g\,f\,k$, is less than the internal and opposite angle, which is impossible. Hence then, they do not coincide at these parts. But they do coincide; and consequently they will be coincident at the other parts, in which the angles less than two right are situated. And thus far Ptolemy.

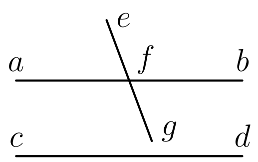
But it is necessary to scrutinize this demonstration, lest perhaps there should be any perverse and captious reasoning in the assumed hypotheses, in those, I say, in which he affirms, that a right line cutting non-coincident right lines, by forming four internal angles, forms the angles at the same parts on each side, either equal to two right, or greater, or less than two right. For the division is not perfect; since nothing hinders our calling those lines non-coincident, which are produced from angles less than two right, denominating, indeed, the two angles at the same parts, greater than two right, but the two at the remaining parts less than two right and not admitting in these, one and the same proportion. But the division being imperfect, the thing proposed is by no means demonstrated. Besides this, also, is not to be passed over in silence against his demonstration, that he does not essentially shew that which is impossible. For it is not because a certain right line cutting parallels, makes the angles at the same parts on each side, greater or less than two right, that an absurdity on this account follows these hypotheses. Nevertheless, because the four angles within the lines which are cut, are equal to four right, on this account each of the hypotheses is impossible; since, if parallel lines are not assumed, yet, when the same hypotheses are assumed, the same consequences will be the result. And such are our animadversions against the demonstration of Ptolemy: for the imbecility of his demonstration appears from what has been said.

Let us now consider those, who say it is impossible that lines produced from angles less than two right, should coincide. For when they have assumed two right lines $a\,b$, $c\,d$, and a right line $a\,c$, falling upon them, and making the two internal angles less than two right, they say it is possible that the right lines $a\,b$, $c\,d$ may be shewn to be non-coincident. For let $a\,c$ be bisected



in e, and cut off from $a\,b$, a part $a\,f$, equal to $a\,e$: but from $c\,d$, a part $c\,g$, equal to $e\,c$. It is manifest, therefore, that the right lines $a\,f$, $c\,g$, will not coincide in the points f and g. For if they coincide, these two in the triangle will be equal to $a\,c$, which is impossible. Let again $f\,g$ be connected, and bisected in h, and cut off equal parts. These, therefore, will not coincide on the same account, and this will be the case, in infinitum, by connecting the non-coincident points, and bisecting the connecting line, and by cutting from the right lines, lines equal to the halves of the connecting lines; for by this means they say, that the right lines $a\,b$, $c\,d$ will never coincide. To such as these we reply, that they indeed affirm that which is true, but not so much as they imagine. For it is not true that the point of coincidence is simply determined by this means, nor is it true that the lines by no means coincide. Thus, when the angles $b\,a\,c$, and $d\,c\,a$, are determined, the lines $a\,b$, and $c\,d$, will not coincide in the points f and g, yet nothing hinders their coinciding in the points k and l, though $f\,k$ and $g\,l$ should be equal to $f\,h$, and $h\,g$. For when $a\,k$ and $c\,l$ coincide, the angles $k\,f\,h$, $l\,g\,h$, will not remain the same, and a certain part of the right line $f\,g$, will be left external to the right lines $a\,k$ and $c\,l$; and so again the two lines $f\,k$, and $g\,l$, are so much greater than the base, as the interior parts of the right line $f\,g$, which they intercept. Besides this also is to be said to such as affirm the non-coincidence of lines extended from angles less than two right, that they destroy what they are unwilling to destroy. For let the same description be given. Wherefore, therefore, is it possible, or impossible to connect a right line from the point a, to the point g? For if it be impossible, besides destroying the fifth petition, they also destroy that which says, *that a right line may be drawn from every point to every point*: but if possible let it be connected. Because, therefore, the angles $f\,a\,c$, $g\,c\,a$, are less than two right, it is manifest that the angles also $g\,a\,c$, $g\,c\,a$, are much less than two right. The right lines, therefore $a\,g$, $c\,g$, will coincide in the point g, being produced form angles less than two right. Hence, it is not possible to affirm indeterminately, that lines produced from angles less than two right, will not coincide. It is however manifest, that some right lines produced from angles less than two right will coincide, though the present discourse seems to investigate this in all. For it may be said, that when the diminution of two right lines is indefinite, the lines will remain non-coincident according to such a diminution: but will coincide according to another less than this. But he who desires to behold a demonstration of this affair, must be informed that it is requisite for this purpose to pre-assume such an axiom as is employed by Aristotle[173] in proving the world to be finite, viz. *If from one point two right lines forming an angle are produced in infinitum, the distance of the lines infinitely produced will exceed every finite magnitude.* For he shews that when infinite right lines are produced from the centre to the circumference, the interval also contained between them will be infinite: since, if it be only finite, it is possible that the distance may be increased; and on this account the right lines will not be infinite. Right lines, therefore, infinitely produced, are distant from each other by an interval greater than every finite magnitude.

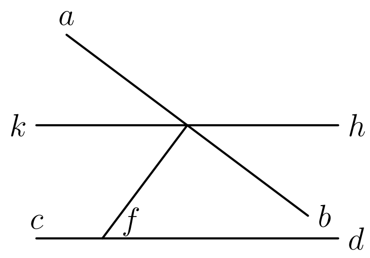
This being pre-supposed, I say that if any right line cuts the one of parallel right lines, it will also cut the other. For let $a\,b$ and $c\,d$ be parallels, and let the right line $e\,f\,g$ cut $a\,b$. I say that it will also cut $c\,d$. For since there are



two right lines, which are produced infinitely from the point f, viz. $b\,f$, and $f\,g$, they shall have a distance greater than every magnitude. Hence, they shall exceed the quantity of the interval contained between the parallel lines. Since, therefore, their distance from each other is greater than that of the

[173]In lib. i. de Caelo, tex. 35.

parallels, $f\,g$ shall cut $c\,d$. But this being demonstrated, we can exhibit the thing proposed in a consequent order. For let there be two right lines $a\,b$, $c\,d$, and let a right line $e\,f$, fall upon them, making the angles $b\,e\,f$, $d\,f\,e$, less than two right. I say that the right lines will coincide in those parts, in which the angles less than two right are situated. For since the angles $b\,e\,f$,



$d\,f\,e$, are less than two right, let the angle $h\,e\,b$, be equal to the excess of two right angles above these angles, and produce $h\,e$, to the point k. Because, therefore, a right line $e\,f$, falls upon the right lines $h\,k$, $c\,d$, and makes the internal angles equal to two right, viz. the angles $h\,e\,f$, $d\,f\,e$, the right lines $h\,k$, $c\,d$, are parallel; and $a\,b$ cuts $k\,h$. It will therefore also cut $c\,d$, by the assumption previously exhibited. Hence, the right lines $a\,b$, $c\,d$, will coincide in those parts, in which the angles less than two right are situated. And on this account the thing proposed, is evinced.[174]

[174]Clavius and Simson have employed a multitude of propositions in the demonstration of this petition; but their demonstrations fall far short in my opinion of the elegance of the present.