The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 24

Transcribed by David R. Wilkins

August 2020

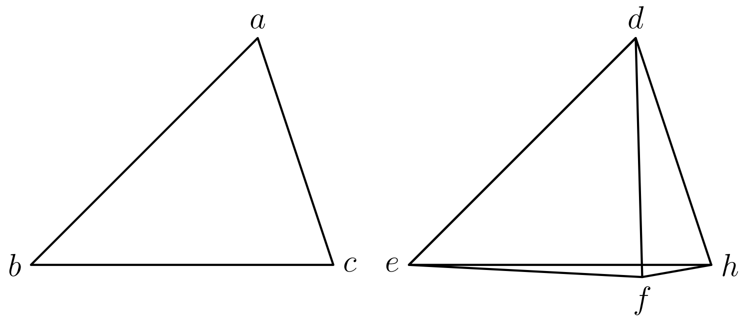
[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 129–136 (1792).]

PROPOSITION XXIV. Theorem XIV.

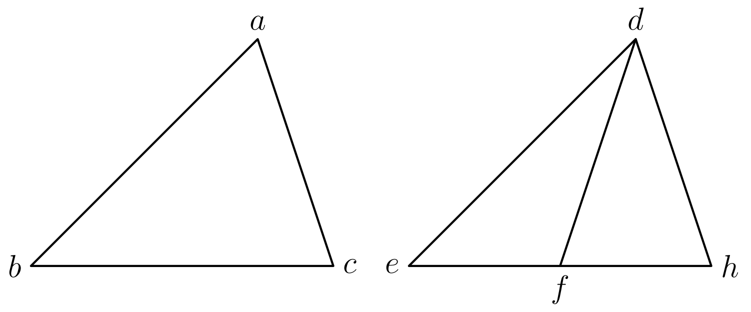
If two triangles have two sides equal to two, each to each, but the one angle contained by the equal right lines greater than the other: they shall also have the base of the one greater than the base of the other.

Euclid again passes on to theorems, and speaks concerning inequality in two triangles, in a mannar similar to his discourse concerning equality. For supposing two triangles, having two sides equal to two, each to each, he sometimes places the vertical angle equal in each, and sometimes unequal; and he proceeds in a similar manner with respect to the base. Besides this, he demonstrates that the equality of the bases is consequent to the equality of the vertical angle, and that the equality of the vertical angles, is consequent to the equality of the bases: but now he shows that the inequality of the one, follows the inequality of the other. The present theorem, therefore, is opposite to the fourth: for that, indeed, supposes the vertical angles of the triangles equal, but this supposes them unequal. And that demonstrates the equality of their bases; but this proves them unequal, in the same manner as their angles. It precedes, however, the following theorem: for that deduces its proof of inequality from the bases to the angles subtending the bases: but this, on the contrary, reasons from the angles to the bases, which are under the angles. Hence it is, after this manner, the converse of its consequent proposition, but opposite to the eighth theorem. For the one from the equality of the bases, deduces the equality of the vertical angles, but the other from the inequality of the bases, shews that the vertical angles are unequal. It is, however, common to these four (two of which are conversant with equality, I mean the fourth, and the eighth, but two about inequality, the present and the following; and two begin from angles, viz. the fourth, and the object of investigation in the present, but two from bases, viz. the eighth and the following proposition); it is common, I say, to all these four, as well as to the fourth and the eighth, as to the twenty-fourth and twenty-fifth, to have two sides equal to two, each to each. For these being unequal, all enquiry is superfluous, and subject to deception. And thus much for a universal speculation concerning the present theorem.

But let us now consider the construction of the elementary institutor, and add to it where deficient. For Euclid receiving two triangles $a\,b\,c$, $d\,e\,f$, having the sides $a\,b$, $a\,c$, equal to the sides $d\,e$, $d\,f$, each to each, and the angle at the point a, being greater than the angle at the point d, and willing to shew that the base $b\,c$, is greater than the base $e\,f$, on the right line $e\,d$, and at a point in it d, constitutes an angle $e\,d\,h$, equal to the angle at the point a.[164] For the

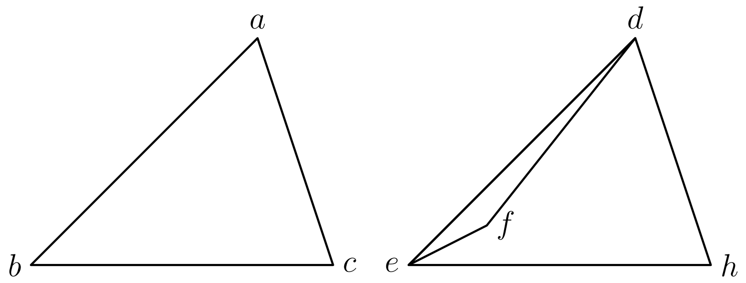


angle at the point a, is greater than the angle at the point d, and he connects $d\,h$, equal to $a\,c$. The right line, therefore $e\,h$, produced to the point h, either falls above, or upon, or beneath the line $e\,f$. The institutor of the Elements, indeed, considers it as lying above the line. But let it be upon the right line. Again, therefore, we may exhibit the same. For the two $a\,b$, $a\,c$, are equal to



the two $d\,e$, $d\,h$, and they contain equal angles. Hence the base $b\,c$, is equal to the base $e\,h$. But $e\,h$ is greater than $e\,f$; and on this account $b\,c$ is greater than $e\,f$. Again, let it be placed beneath $e\,f$. Connecting, therefore, $e\,h$, we must say, that since $a\,b$, $a\,c$, are equal to $d\,e$, $d\,h$, and they comprehend equal angles, $b\,c$ is also equal to $e\,h$. Because, therefore, within the triangle $d\,e\,h$, two right lines $d\,f$, $f\,e$ are constructed on the side $d\,e$, they are less than the external sides. But $d\,h$, is equal to $d\,f$: for it is equal to $a\,c$. Hence $h\,e$ is greater than $e\,f$: But $h\,e$ is equal to $b\,c$. And therefore, $b\,c$ is greater than $e\,f$. The theorem, therefore, is exhibited according to every position.

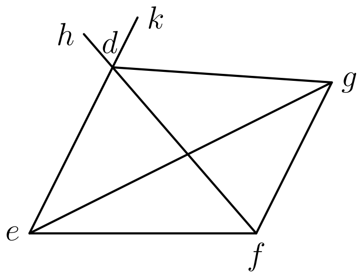
164[DRW—The diagrams illustrating the cases of this proposition have been drawn not as they appear in the 1792 edition, but rather so as to ensure that $d\,f$ is indeed equal to $d\,h$, and thus to $a\,c$, thereby matching the statement of this proposition.]

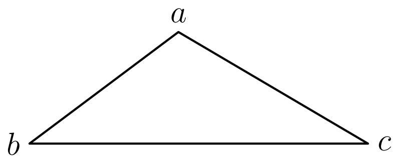


Why then, as is the fourth theorem, he at the same time demonstrated that the areas of the triangles are equal, does he not add in the present, that besides the inequality of the bases, the areas also are unequal? Against this doubt we must say, that there is not the same proportion in equal, as in unequal angles and bases. For when the angles and bases are equal, the equality also of the triangles follows: but when they are unequal, it is not necessary that the inequality of the areas should be consequent: since the triangles may as well be equal, as unequal; and that may be greater, and likewise less, which contains the greater angle, and the greater base. On this account, therefore, the institutor of the Elements leaves the comparison of the triangles; to which we may add, that the contemplation of these, requires the doctrine of parallels.

But if it be requisite, that anticipating things which are afterwards exhibited, we at present make a comparison of areas, we must say, that if the angles a, d are equal to two right, the triangles may be shewn to be equal: but when they are greater than two right, the lesser triangle will be that which contains the greater angle; and when they are less than two right, this will be the case with the greater triangle. For let the construction in the element be given[165], and produce $e\,d$, $f\,d$, to the points k, h; and let us suppose the angles $b\,a\,c$, $e\,d\,f$, equal to two right. Because, therefore, the angle $b\,a\,c$, is equal to the angle $e\,d\,g$, the angles $e\,d\,g$, $e\,d\,f$ are equal to two right. But the angles $e\,d\,g$, $k\,d\,g$, are also equal to two right. Let the common angle $e\,d\,g$

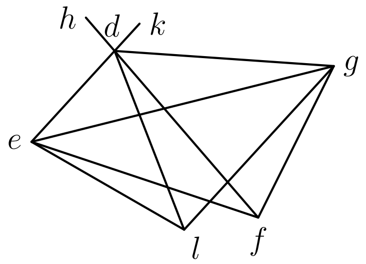
[165] [DRW—The construction referred to by Proclus is that of Euclid's proof of Proposition 24 of Book I, in the case explicitly considered by Euclid. Thus, in the triangles $a\,b\,c$ and $d\,e\,f$, the sides $a\,b$ and $a\,c$ are equal to $d\,e$ and $d\,f$ respectively, but $a\,c$ is greater than $d\,f$. Moreover, in this case of Euclid's proposition, if a point g is taken so that $d\,g$ and $e\,g$ are equal to $a\,c$ and $b\,c$ respectively, then the points d and f lie on opposite sides of the line $e\,g$. Proclus shows that if the angles $b\,a\,c$ and $e\,d\,f$ sum to two right angles then the triangles $e\,d\,g$ and $f\,d\,g$ are equal in area. The proof is included in the commentary of an-Nairīsī on Proposition 38, attributed there to Heron, and accordingly Heath includes a statement and proof of the result, and the corresponding results when the sum of the angles $b\,a\,c$ and $e\,d\,f$ is less than, or greater than, two right angles in his commentary on Proposition 38 of Book I if the *Elements*.]





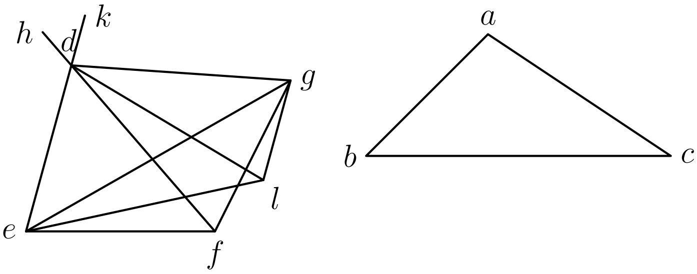
be taken away, and the remainder $e\,d\,f$, will be equal to the remainder $k\,d\,g$. But $e\,d\,f$ is equal to $h\,d\,k$; for they are vertical angles. Hence, the angle $k\,d\,g$, is equal to the angle $h\,d\,k$. And because the angle $g\,d\,h$, is external to the triangle $g\,d\,f$, it is equal to the two internal and opposite angles at the points g and f. But these angles are equal to each other, because $d\,g$ is equal to $d\,f$. Hence, the angle $g\,d\,h$, is double of the angle at the point g, and of the angle at the point f. The angle, therefore, at the point g, is equal to the angle $g\,d\,k$, and they are alternate; and consequently $d\,e$ is parallel to $f\,g$. The triangles, therefore $g\,d\,e$, $f\,d\,e$, are upon the same base $d\,e$, and between the same parallels $d\,e$, $g\,f$; and are consequently equal. But the triangle $g\,d\,e$, is equal to the triangle $a\,b\,c$; and so the triangle $d\,e\,f$, is not unequal to the triangle $a\,b\,c$. And here you may observe, that we require three theorems belonging to the doctrine of parallels; one, indeed, affirming, *that the external angle of every triangle is equal to the two internal and opposite angles*: but the other, *that if a right line falling upon two right lines, makes the alternate angles equal, the right lines are parallel*; and the third, *that triangles constituted upon the same base, and between the same parallels, are equal*, which the institutor of the Elements, also knowing, omits the comparison of triangles.

But let the angles $b\,a\,c$, $e\,d\,f$, be greater than two right, and let the same things be constructed. Because, therefore, the angles $b\,a\,c$, $e\,d\,f$, i.e., the angles $e\,d\,g$, $e\,d\,f$, are greater than two right, but the angles $e\,d\,g$, $g\,d\,k$, are equal to two right, by taking away the common angle $e\,d\,g$, the angle $e\,d\,f$, is greater than the angle $g\,d\,k$. Hence, the angle $g\,d\,h$, is more than double of the angle $g\,d\,k$; and so the angle $g\,d\,k$, is less than the angle at the point g. Let $g\,d\,k$ be placed equal to $d\,g\,l$, and let $e\,l$ and $d\,l$, be connected: $g\,l$, therefore, is parallel to $d\,e$; and hence, the triangles $g\,d\,e$, $l\,d\,e$ are equal. But the triangle $l\,d\,e$, is less than the triangle $f\,d\,e$. The triangle $g\,d\,e$, is less



than the triangle $f\,d\,e$. But the triangle $g\,d\,e$, is equal to the triangle $a\,b\,c$; and hence, the triangle $a\,b\,c$, is less than the triangle $f\,d\,e$, viz. is less than the triangle which contains the greater angle [*sic.*].

In the third place, let the unequal angles be less than two right, and let the same things be constructed. Because, therefore, the angles $e\,d\,g$, $g\,d\,k$, are equal to two right, by taking away the common angle $e\,d\,g$, the whole $g\,d\,h$, is less than the double of $g\,d\,k$. But it is double also of the angle at the point g. Hence, the angle $g\,d\,k$, is greater than the angle at the point g. Let the angle $d\,g\,l$ be placed equal to the angle $g\,d\,k$, and let $g\,l$ coincide with $e\,l$, in the point l, and connect $d\,l$. Hence, $g\,l$ is parallel to $d\,e$; and consequently



the triangles $g\,d\,e$, $l\,d\,e$, are equal to each other. But the triangle $l\,d\,e$, is greater than the triangle $f\,d\,e$; and the triangle $g\,d\,e$, is equal to the triangle $a\,b\,c$. Hence, the triangle $a\,b\,c$, is greater than the triangle $d\,f\,e$. It is shewn, therefore, that the triangle $a\,b\,c$ is both equal to, and is also greater and less than the triangle $d\,e\,f$, the angles at the points a and d, being either equal to, or greater or less than[166] [*sic.*] two right. And thus, all the hypothesis may be accomplished. For what if the angle at the point a, should be one right, and the half of a right angle, but the angle at the point b, the half of one right, would not those two angles be equal to two right? But what if the angle at the point a should be one right, and the half of a right, but

[166] [DRW—The triangle for which the equal sides contain the greater angle has the lesser area in the case where the angles contained by the equal sides sum to more than two right angles, but has the greater area in the case where those angles sum to more than two right angles.]

the angle at the point b, two thirds of one right, would they not be greater than two right angles? And lastly, if the angle at the point a, should be one right, and the half of a right angle, but the angle at the point b, a third part of a right angle, would they not be less than two right, and the angle a be greater than the angle d? All these comparisons, therefore, are produced by the assistance of parallels; and hence, they are necessarily not found in the present elementary institution.