The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 15

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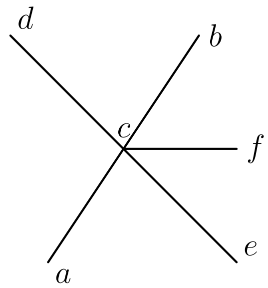
[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 96–101 (1792).]

PROPOSITION XV. Theorem VIII.

If two right lines cut one another, they will form the angles at the vertex equal.

We must call *successive* angles different from such as are *vertical.* For these last originate from the section of two right lines: but the former from the mere dissection of the one by the other. Thus, if a right line remaining itself without section, but cutting another in its extremity, forms two angles, we denominate these *successive* angles. But if the two right lines mutually cut each other, they form *vertical angles.* And they are so called, because they have their vertices conjoined in the same point. But their vertices are the points, at which the planes, while they are contracted, form angles. This, therefore, is what the present theorem evinces, that when two right lines mutually cut each other, the vertical angles are equal. And it was first invented (according to Eudemus) by Thales: but was thought worthy of a demonstration producing science by the institutor of the Elements. But it is not exhibited from all the particulars requisite to a perfect proposition. For construction is wanting in the present theorem: but demonstration, which must be necessarily inherent, depends on the thirteenth theorem. But he uses two axioms, one of which is, *that things equal to the same, are equal among themselves*: and the other, *if from equal things equals are taken away, the remainders will be equal.* The theorem, indeed, of Euclid, is manifest, but another such is converted to the present theorem. *If to any right line, and at a point in it, two right lines, not assumed towards the same parts, make the vertical angles equal, those right lines shall be in a direct position to each other.* For let there be a certain right line $a\,b$, and any point in it c, and at the point c, let two right lines $c\,d$, $c\,e$, not towards the same parts be assumed, forming equal angles $a\,c\,d$, $b\,c\,e$. I say that $c\,d$, $c\,e$, are in a right line. For since the right line $c\,d$, insists upon the right line $a\,b$, it forms angles equal to two right, i. e. $d\,c\,a$, $d\,c\,b$. But the angle $d\,c\,a$, is equal to the angle $b\,c\,e$. Therefore, the angles $d\,c\,b$, $b\,c\,e$, are equal to two right. Because, therefore, to a certain right line $b\,c$, and at a point in it c, two consequent right lines $c\,d$, $c\,e$, not placed towards the same parts, form the successive angles equal to two right, those right lines $c\,d$, $c\,e$, are in a direct position to each other. The converse, therefore, to the present theorem, is exhibited. But the geometrician seems to have neglected this, because it is easy to evince its truth, by the same

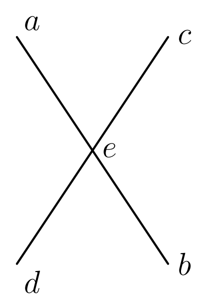


method of deduction to an impossibility as we employed in exhibiting the fourteenth proposition. For the same things being supposed, I say that the right line $c\,d$, is in a direct position to $c\,e$. For if it be not, let $c\,f$ be taken in a right line with $c\,d$. Because therefore, two right lines $a\,b$, $d\,f$, intersect each other, they will form the angles at the vertex equal. Hence, the angles $a\,c\,d$, $b\,c\,f$, are equal. But $a\,c\,d$, $b\,c\,e$, were also equal. The angle, therefore, $b\,c\,e$, is equal to the angle $b\,c\,f$, the greater to the less, which is impossible. Hence, no other right line, besides $c\,d$, is in a direct position to $c\,e$. The right lines, therefore, $c\,d$, $c\,e$, are in a direct position to each other, the angles at the vertex being supposed equal. Since then, there is the same demonstration which was preassumed in the fourteenth theorem, would it not have been superfluous to have produced this conversion? But for the sake of exercise, we have proved it as well by a deduction to an impossible, as by an ostensive method. However, this fifteenth theorem seems to rest upon the similitude of the parts of right lines, and their situation in their extremities. Because lines with these conditions, and mutually cutting each other, must necessarily possess similar inclinations on both sides to each other. Since circumferences, and universally non-right lines cutting one another, do not necessarily form the vertical angles equal, but sometimes equal, and sometimes unequal. For if two equal circles cut each other through the centres, or even not through the centres, they will form the lunular angles at the vertex equal: but not likewise the remaining angles, viz. those on both sides concave, and on both sides convex, but the one will be greater than the other. But in right lines, the situation in the extremities, causes the distance of one segment, to be equal to the distance of another.

COROLLARY.

From hence it is manifest that if two right lines cut each other, they will make four angles equal to four right.

Corollary[152] is one of the geometrical appellations, but it has a twofold signification. For they denominate corollaries, whatever theorems are proved together with the demonstrations of others, becoming as it were the unexpected gain and emolument of the investigator: and likewise, whatever is the object of enquiry, but is indigent of invention, and is neither investigated for the sake of generation alone, nor of simple contemplation. For that the angles at the bases of isosceles triangles are equal, it is requisite to contemplate, and the knowledge of things in existence is of this kind. But to bisect an angle, or constitute a triangle, to cut off, or place an equal right line, all these demand that something may be performed. And again, to find the centre of a given circle, or two commensurable magnitudes being given to find their greatest common measure, with every thing of this kind, are, after a manner, situated between problems and theorems. For neither is the origin of *objects of enquiry* inherent in these, nor contemplation alone, but invention, Since it is requisite to place the object of enquiry conspicuously and before our eyes. Such then are whatever corollaries Euclid wrote, for he constructed a book of corollaries. But we must now omit to speak of corollaries of this kind. However, such as occur in the elementary institution, appear at the same time with the demonstration of other things, but they themselves are not principally investigated, as is evident in that which is proposed at present. For the design of the proposition is to enquire whether if two right lines mutually cutting each other, the angles at the vertex are equal. But whilst this is evinced, it is at the same time demonstrated, that the four angles which are formed, are equal to four right. For when we say let there be two right lines ab, cd, cutting each other in the point e: because ae stands on cd, it makes the successive angles equal to two right. And again, because be



stands upon cd, it also makes the successive angles equal to two right; then together with the object of enquiry we demonstrate, that the angles about the point e, are equal to four right. A corollary, therefore, is a theorem, unex-

[152] [DRW—in discussions of Greek mathematics, it is customary to use the term *porism*, rather than *corollary*, in this context.]

pectedly emerging from the demonstration of another problem, or theorem. For we seem to fall upon corollaries, as it were, by a certain chance; and they offer themselves to our inspection, without being proposed, or investigated by us. Hence, we assimilate these also to gains. And perhaps those skilled in mathematical concerns, have imposed on them this appellation, shewing the vulgar, who rejoice in apparent gain, that these are the true gifts of divinity, and true gains, and not the objects of their sordid estimation. For this indeed produces that faculty resident in our nature, and adds the prolific power of science, to principal enquiries, manifesting the copious riches of theorems. And such is the property of corollaries.

But they are to be divided in the first place, according to sciences. For of corollaries, some are geometrical, but others arithmetical. Thus the present corollary is geometrical: but that which is added at the end of the second theorem of the seventh book of the arithmetical elements, is arithmetical. But afterwards they must be divided according to the principal objects of enquiry. For some things are consequent to problems, but others to theorems. Thus the present is consequent to a theorem: but that which is placed in the second of the seventh book, is consequent to a problem. But in the third place, they must be divided according to their ostensions. For some are exhibited, together with ostensive methods, but others together with deductions to an impossible. Thus the present is shewn by a direct ostension: but that which is exhibited in the first of the third book, appears, together with a deduction to an impossible. But corollaries may also be divided in many other modes, but these may suffice our present purpose. The present corollary, however, teaching us that the place about one point is distributed into angles equal to four right, is subservient to that admirable theorem, which shews that the following three multangles about one point, can alone fill place, viz. the equilateral triangle, the quadrangle, and an equilateral, and equiangular sexangle. But the equilateral triangle must be six times assumed; since six two-thirds, form four right angles. But the sexangle must be three times formed; for every sexangular angle is equal to one right, and a third part of a right. And a quadrangle must be four times assumed: for every quadrangular angle is right. Hence, six equilateral triangles conjoined according to their angles, fill four right angles, as also three sexangles, and four quadrangles. But all other multangles, however composed, according to angles, are either deficient from four right, or exceed four right angles[153]; while these alone, according to the aforesaid numbers, are equal to four right.

[153]That no other figure besides these can fill place, will be evident, if its angle, when found, is multiplied by any number; for, as Tacquet well observes, it will always either exceed, or be deficient from four right angles. For a more particular demonstration of this admirable theorem, see Tacquet Elementa Geometriæ, p. 88.

And this theorem is Pythagoric. But by the present corollary, if even more than two right lines should cut each other in one point, as for instance, three or four, or any other number, all the angles which they form, may be shewn to be equal to four right. For they will vindicate to themselves the place of four right angles. But it is manifest that the angles always become double to the number of right lines. And thus two right lines intersecting each other, there will be four angles equal to four right: but from the intersection of three lines, there will be six angles; and from four, eight, and so on, in infinitum. For the multitude of the right lines is always doubled: but the angles increase according to multitude, and are diminished according to magnitude, because it is the same four right angles, which is perpetually divided.