The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 9

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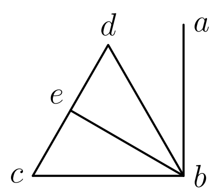
[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 70–77 (1792).]

PROPOSITION IX. PROBLEM IV.

To bisect a given rectilineal angle.

Our author mingles theorems with problems, and connects problems with theorems, and through both completes the whole of his elementary institution, comparing as well subjects as the *symptoms* subsisting about subjects themselves. Since, therefore, he had shewn in the preceding propositions, both in one triangle, from the equality of the sides, the consequent equality of the angles, and the contrary: and in a similar manner in two triangles, with this exception, that the mode of conversion in one and two triangles is different, he now passes to problems, and orders us to bisect a rectilineal angle. And it is manifest, that the angle here is given according to form: for it is called right-lined, and not of any kind whatever. Indeed, we cannot bisect every angle by the elementary institution; since it is doubtful whether every triangle can be bisected. For, perhaps you may doubt whether it is possible to bisect a cornicular angle. But the ratio of the section is also distinguished in this problem, and this again not in vain. For to divide an angle in any given ratio, transcends the present construction: as, for example, into three, four, or five equal parts. Indeed, to trisect a right angle is possible, by employing a few of the propositions which are afterwards delivered[145]: but this cannot be effected in an acute angle, without passing on to other

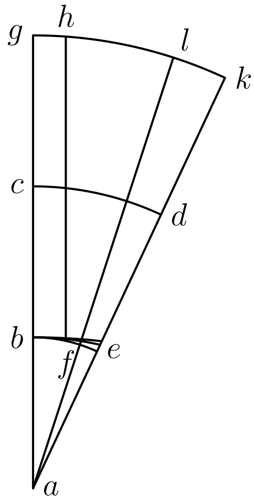
[145]This too may be easily effected by means of the first problem, and the present. Thus let $a\,b\,c$ be a right angle, which it is required to trisect; then, upon the side $a\,b$, describe an equilateral triangle $c\,d\,b$, and bisect the angle $d\,b\,c$, and the angles $a\,b\,d$, $d\,b\,e$, $e\,b\,c$ shall be equal.



For the angle $d\,b\,c$, is one third of two right angles, or two thirds of one right, and consequently the angle $d\,b\,a$, is one third of a right angle; and this is equal to $d\,b\,e$, the half of $d\,b\,c$. Therefore they are all equal.

lines of a mixed species.[146] And this is manifested by the geometricians who propose to trisect a given rectilineal angle. For *Nichomedes*, indeed, from conchoidal lines, the origin, order and symptoms of which, he delivers, as he was the inventor of their properties, trisects every right-lined angle. But

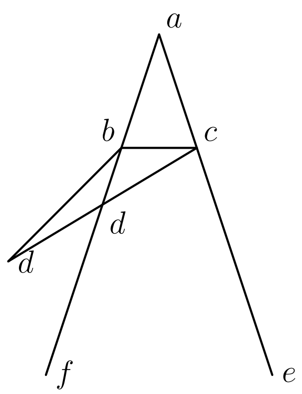
[146]The method of dividing an angle in any given ratio, by means of a right line and circle only, seems to have been entirely unknown to the ancients, as well as to the moderns. However, the author of this translation presumes he has discovered the means of solving this arduous problem; and that such as admit the truth of his demonstration respecting the quadrature of the circle in page 56 of his Dissertation, vol. I. of this work, must necessarily subscribe to the following method of dividing an angle in any required proportion. Let there be an acute angle given $g\,a\,k$, which it is required to divide in the ratio of the right line $a\,c$ to $c\,g$. Bisect $a\,c$ in b, and from the centre a, with the radius $a\,b$, describe the arch $b\,e$, and with a radius equal to $a\,c$, describe an arch touching $b\,e$ in the point b.



Likewise with a radius double to $a\,c$, describe another tangent arch at the point b, and with a radius equal to $a\,g$, a tangent arch at the same point, according to the figure; and lastly, let the arches $c\,d$, $g\,k$, from the centre a be drawn. Then $\frac{1}{2}$ of the arch $b\,e$ shall be equal to $\frac{1}{4}$ of $c\,d$, and to $\frac{1}{8}$ of the arch similarly placed, described with a radius the double of $a\,c$, as is well known. Bisect then $b\,e$ in f, and make each of its two next tangent arches at b, equal to $b\,f$, which is easily done, from what has been already observed; and through the points of equality describe a circle, this (by the theorem in page 56 of our Dissertation), shall cut off some part of the tangent arch described with the radius $a\,g$, equal to $b\,f$, or the fourth part of $c\,d$. Hence, a part in the arch $g\,k$, may easily be taken equal to $c\,d$, which let be $g\,l$, and drawing the right line $a\,l$, the angle $g\,a\,l$, shall be to $l\,a\,k$, as $a\,c$ to $c\,g$, which was required to be done.

The same construction will serve for the division of a right angle in any given ratio, as is evident; and if the given angle be obtuse, the problem may be solved by a two-fold operation, that is, by bisecting the obtuse angle, and dividing either of the equal sections in the given ratio; for when this is effected, the whole angle may be easily divided in the same proportion. Hence, too, a right line may be speedily obtained equal to a given arch of a circle.

others effect this from the quadrantal lines of *Hippias* and *Nichomedes*, by employing mixt quadrantal lines. Others, again, being incited from the Helices of *Archimedes*, divide a given rectilineal angle, in a given ratio. But the consideration of these, because difficult to learners, we shall for the present omit; as it will, perhaps, be more convenient to examine this in the third book[147], where the institutor of the Elements bisects a given circumference. For there the same mode of enquiry presents itself with respect not only to bisection, but also trisection; and the ancients endeavored, by employing the same lines, to divide every circumference into three equal parts. With great propriety, therefore, he who only mentions a right line and a circumference, alone bisects a right angle and a circumference. But conceiving that the species composed of these, through mixture, are difficult to explain and enumerate, without a curious examination, he omits all such enquiries as involve mixt lines in their consideration, and proposes to investigate in first and simple forms alone, such things as can either be produced or considered from these. And such, indeed, is the proposition of the present problem, *to bisect a given right lined angle*. For in the construction of this he uses one petition, and the first and third problem: but in the demonstration he employs the eighth theorem alone. Since problems entirely require demonstration (as we have already observed[148] and through this they obtain the power of producing science. But perhaps, some may oppose the geometrician, by asserting that an equilateral triangle may be constituted by him, not having its vertex within the two right lines, but either upon, or external to each, and that this may be manifested by the elements. For let there be an angle $b\,a\,c$, which it is required to bisect. Then let $b\,a$ be taken equal to $a\,c$, and let $b\,c$ be connected, and upon it, let an equilateral triangle $b\,c\,d$ be constructed. The



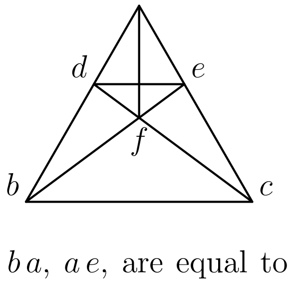
point d, therefore, is either within the right lines $a\,b$, $a\,c$, or upon $a\,b$, or

[147]In the 30th Prop.

[148]See Chap. 8. Book 2d.

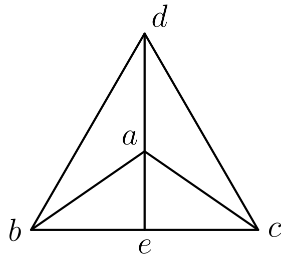
$a\,c$, or external to both. Now the institutor of the Elements assumes them within; and hence, those who oppose the demonstration, will say the point is either placed on one of the right lines, or external to both. Let the point d then be placed on the line $a\,b$, so that the triangle $b\,c\,d$ may be equilateral: $d\,b$, therefore, is equal to $d\,c$, and the angles at the base $c\,b\,d$, $b\,c\,d$, are equal. Hence, the whole $b\,c\,e$, is greater than the angle $c\,b\,d$. Again, because $a\,b$, $c\,a$, are equal, the triangle $a\,b\,c$, is isosceles, and the angles under the base $b\,c$, will be equal. The angle, therefore, $b\,c\,e$, is equal to the angle $c\,b\,d$. But it is also greater, which is impossible. Hence the vertex of the equilateral triangle cannot be in the right line $a\,b\,d$. In like manner we may shew that it cannot be in the right line $a\,c\,e$. Let it therefore, if possible be placed externally. Because, then $b\,d$ is equal to $c\,d$, the angles at the base are equal, viz. $b\,c\,d$, and $c\,b\,d$. Hence, the angle $b\,c\,d$, is greater than the angle $c\,b\,f$. Much more, therefore, is the angle $b\,c\,e$, greater than $c\,b\,f$: but it is also equal, because these angles are under the base $b\,c$, of an isosceles triangle $a\,b\,c$, and this is impossible. Hence, the point cannot fall in these parts external to the two right lines; and it may be similarly shewn that this is impossible in other parts. Here too you may again observe, that we destroy objections by using the second part of the fifth proposition, *that the angles under the base of an isosceles triangle are equal.* And this is what we have previously observed, that many things opposing science, are shewn to be debile, and easy of confutation, by the assistance of this theorem; and that such is the utility it affords to geometry.

But if any one should say that there is no place under the base, and yet that it is requisite to constitute the equilateral triangle at the same parts, in which the lines $b\,a$, $a\,c$, are situated; it will be necessary that the lines which are constituted should either coincide with $b\,a$, $a\,c$, if they also are equal to the base $c\,b$: or that they should fall external to them, if they are less than the base $b\,c$: or within, if $b\,a$, $a\,c$, are greater than $b\,c$. Let them, in the first place, coincide, and let $b\,a\,c$ be an equilateral triangle, and let there be taken in the side a, b, the point d, and make $a\,e$ in the side $a\,c$, equal to $a\,d$, and connect the lines $d\,e$, $b\,e$, $c\,d$, $a\,f$. Because, therefore, $a\,b$ is equal to



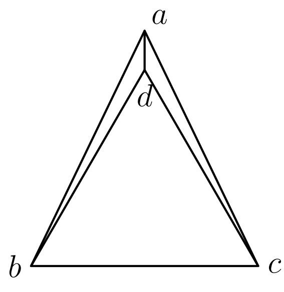
$a\,c$, and $a\,d$ to $a\,e$, the two $b\,a$, $a\,e$, are equal to the two $c\,a$, $a\,d$, and they comprehend the same angle. Hence, they are all equal to all, and the angle dbe, is equal to the angle ecd. But db is also equal to ec, and be to cd. All, therefore, are equal to all. Hence, the angle deb, is equal to the angle edc: for they subtend equal sides. And df is equal to ef, (by the sixth.) Because, therefore, ae is equal to ad, and af is common, and the base df, is equal to the base ef, the angle dae is bisected, which was required to be done.

But if the sides of the equilateral triangle fall external to the right lines ba, ac, let them be bd, dc, and having connecting da, let it be produced to the point e. Because, therefore bd, dc, are equal, but da is common, and the



bases ba, ac, are equal, the angle, also, bda, (by the eighth) is equal to the angle cda. Again bd, dc, are equal, and de is common, and they contain equal angles as we have shewn, the base also be, is equal (by the fourth) to the base ec. Because, therefore, ab is equal to ac, and ae is common, the angle, also bae, is equal to the angle cae, which was to be shewn.

But if the sides of the equilateral triangle fall within the right lines ab, ac, as bd, dc, let again ad be connected. Because, therefore, ba, is equal to ac,



and ad is common, but the base bd, is equal to the base cd, hence, the angle bad (by the eighth) is equal to cad. The angle, therefore, at the point a, is bisected, in whatever manner the equilateral triangle may be constituted. And having thus summarily spoken concerning these, we shall now proceed to the following theorems, only adding, that the given angle may be given in a four-fold respect. *In position*, as when we say *to this right line, and to this* *point to place an angle*: for after this manner it is given. But *in form*, as when we call the angle right, or acute, obtuse, right-lined, or mixed. And *in proportion*, as when we call it double, or triple, greater, or less. And lastly, *in magnitude*, as when we call it the third part of a right angle. But the present angle is only given in form.