# The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 6

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[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 55–60 (1792).]

## PROPOSITION VI. Theorem III.

If two angles of a triangle be equal to each other, the sides also which subtend the equal angles, shall be equal to one another.

The present theorem exhibits these two properties of theorems, *conversion*, and a *deduction to an impossibility.* For it is converted, indeed, in the preceding theorem, but its certainty is evinced by a deduction to an impossibility. It is requisite, therefore, to speak of each, whatever belongs to the present treatise. One kind of *conversion* then, among geometricians, is denominated principally and properly, when the conclusions and hypotheses alternately receive theorems; so that the conclusion, of the former becomes hypothesis in the latter; and the hypothesis is inferred as the conclusion. As *that the angles at the base of an isosceles triangle are equal.* For here the isosceles triangle is the *hypothesis*: but the *conclusion*, the equality of the angles at the base. And *that where the angles at the base are equal, the triangles are isosceles*, which the present 6th theorem affirms. For here the equality of the angles at the base is the *hypothesis*; but the *conclusion*, the equality of the sides subtending the equal angles. But another kind of conversion, is alone according to a certain mutation of composites. For if the theorem be composite, beginning with many hypotheses, and ending in one conclusion, by receiving the conclusion, and one or more of the hypotheses, we infer some one of the other hypotheses as a conclusion. And after this manner the eighth theorem is the converse of the fourth. For the one says, that *equal bases subtend equal sides and angles*: but the other, that *equal sides being placed on equal bases, contain equal angles.* Of which the predication concerning *equal bases* in the latter proposition, is the *conclusion* of the former: but the predication concerning the position of *equal sides*, is one of the previously assumed hypotheses in the former theorem; and the *comprehension of equal angles* is another hypothesis which this fourth proposition contains. In consequence therefore of these two *conversions*, the one which is called the *principle*, is uniform and determinate: but the other is various, advancing into a great number of theorems, and not converting in one, but in many, on account of the multitude of hypotheses, in composite theorems. But oftentimes in that which begins from two hypotheses, there is one which is converted, when the hypotheses are not all determinate, but some of them indeterminate.

It is here, however, requisite to observe, that many false and improper conversions take place. As that *every sexangular is a triangular number*[140]. For the converse is not also true, that *every triangular number is sexangular.* But the reason of this is, because the one is more common, but the other more particular. And one is alone predicated *totally*[141] of the other. But things in which, *that which is primary*, is inherent, and according to which it is received, in these, conversion also follows. And these observations, indeed, were not unknown to those mathematicians, the familiars of Menæchmus and Amphinomus. But of theorems receiving conversion, some are usually called *precedents*, but others *converse.* For when supposing a certain genus, they demonstrate some symptom of its nature, they call this a *precedent* theorem. But when on the contrary, they make the hypothesis a symptom, and the conclusion a genus, they denominate the theorem to which this happens *converse.* As for instance, the theorem which says, *every isosceles triangle has the angles at the base equal*, is a *precedent.* For that is subjoined which precedes by nature. I mean the genus itself, or the isosceles triangle. But that which says, *every triangle possessing two equal angles, has likewise the sides subtending those equal angles equal, and is isosceles*, is a *converse* theorem. For it changes the subject, and its passion, supposing the latter, and from this exhibiting the former. And thus much concerning geometrical conversions.

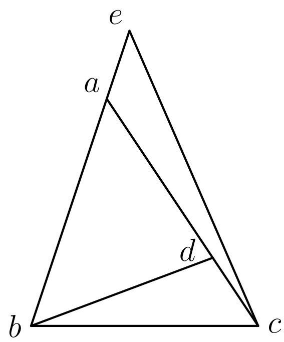
But deductions to an impossibility, entirely end in an evident impossible, the contrary of which is confessed by all. It happens, however, that some of them end in such things as are opposed to Axioms, or Petitions, or Hypotheses; but others in things contradicting prior demonstrations. For the present sixth theorem shews that which happens to be impossible, because it destroys the common conception, affirming that *the whole is greater than its part.* But the eighth theorem falls, indeed, on an impossible, yet not on that endued with a power of destroying a common conception, but that exhibited by the seventh theorem. For what the seventh denies, this affirming exhibits to such as do not admit the object of investigation. But every deduction to an impossibility, which being received, opposes *the thing sought*, and on this hypothesis advances, until it falls upon the explored absurdity, and by this means destroys the hypothesis, coroborates [*sic.*] that which was in-

[140] Triangular numbers, are 1, 3, 6, 10, &c.; and sexangular numbers 1, 6, 15, 28, &c. But concerning their formation, see note to page 95, Vol. I. of this work; by means of which, the truth of this assertion will be evident

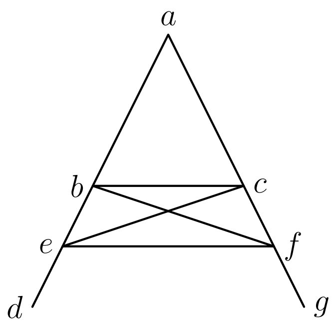
[141] Concerning the meaning of total predication, see page 45 of the Dissertation, Vol. I. of this work.

vestigated from the first. But it is requisite to know, that all mathematical proofs are either *from principles*, or *to principles*, as Porphyry in a certain place affirms. And the proofs *from principles*, are two-fold. For they either emanate from common conceptions, and things self-evident: or from things previously exhibited. But proofs *to principles* are endued with a power of either *establishing* or *destroying principles.* And those, endued with a power of *establishing principles*, are called *resolutions*; and to these compositions are opposed. For it is possible that we may proceed in an orderly method from those principles to the object of investigation; and this is nothing else than composition. But those possessing a power of *destroying principles*, are called *deductions to an impossibility.* For it is the business of this mode to destroy some of the concessions, and objects of investigation. And in this, also, there is a certain ratiocination, though not the same as in resolution. For in deductions to an impossibility, *complexion* is according to the second mode of hypothetical reasonings. As if *in triangle possessing equal angles, the sides subtending the equal angles are unequal; and the whole is equal to its part*: but this is impossible. *In triangles, therefore, possessing two equal angles, the sides subtending the equal angles are equal.* And thus much concerning what is called by geometricians, deduction to an impossibility.

But the institutor of the elements uses *conversion* in the present proposition, for he receives the conclusion of the fifth as a datum, and adds its hypothesis as an object of enquiry: but he employs *deduction to an impossibility*, in the construction and demonstration. But if any should rise up, and assert that it is not necessary by taking a part from $a\,c$ equal to $a\,b$, to make the ablation at the point $c$, but at the point $a$, upon this hypothesis, we shall fall into the same impossibility. For let $a\,b$ be equal to $a\,d$, and having produced $b\,a$, let $a\,e$ be placed equal to $d\,c$. The whole $b\,e$, therefore, is equal to the whole $a\,c$. Let $e\,c$ be connected.




Because, therefore, $a\,c$ is equal to $b\,e$, but $b\,c$ is common, the two are equal to the two, and the angle at the point $b$, is equal to the angle $a\,c\,b$. For so it was established in the hypothesis. All, therefore, are equal to all, by the fourth theorem. Hence the triangle $e\,b\,c$, is equal to the triangle $a\,b\,c$, the whole to the part, which is impossible. But because this also is manifest, it remains that we exhibit the rest of the conversion. For the institutor of the Elements converts the whole sixth theorem from a part of the fifth. But it is requisite to adjoin the remaining conversion. This, then, he receives as an hypothesis, *that the angles at the base of a certain triangle are equal*: but he shews that the triangle is *isosceles*. Let $a\,c\,b$, therefore, be a triangle, and let $a\,b$, $a\,c$, be produced to the points $d\,g$, and let the angles under the base be equal. I say that the triangle $a\,b\,c$, is isosceles. For let there be assumed in the line $a\,d$, the point $e$, and let $b\,e$ be taken equal to $c\,f$; and connect the lines $e\,c$, $b\,f$, $e\,f$.




Because, therefore $b\,e$ is equal to $c\,f$, but $b\,c$ is common, the two will be equal to the two. And the angle $e\,b\,c$, is equal to the angle $f\,c\,b$; for they are under the base. All, therefore, are equal to all, by the fourth theorem. Hence the base $e\,c$, is equal to the base $f\,b$, and the angle $b\,e\,c$, to the angle $c\,f\,b$; and the angle $c\,b\,f$, to the angle $b\,c\,e$: for they subtend equal sides. But the whole angle $e\,b\,c$, was equal to the whole $f\,c\,b$, of which the angle $f\,b\,c$, is equal to the angle $e\,c\,b$. the remainder, therefore, $e\,b\,f$, is equal to the remainder $f\,c\,e$. But $b\,e$ is equal to $c\,f$, and $b\,f$ to $c\,e$, and they contain equal angles. All, therefore, are equal to all. Hence, also, the angle $b\,e\,f$, is equal to the angle $c\,f\,e$. Wherefore, the side $a\,e$, is equal to the side $a\,f$ (for it is shewn by the sixth) of which $b\,e$, is equal to $c\,f$. The remainder, therefore $a\,b$, is equal to the remainder $a\,c$. And hence, the triangle $a\,b\,c$, is isosceles. It is, therefore, as well isosceles, if it possesses angles at the base equal: as if the sides being produced it has the angles under the base equal. Why then did not the institutor of the Elements convert the remaining part? Shall we say it was because the equality of the angles under the base in the fifth theorem, was exhibited for the sake of solving other doubts. But that proving the triangle to be isosceles, from the equality of the angles under the base, neither confers to a principal demonstration, nor to the solution of things investigated, the truth of which is confirmed in the following theorems, and that from the equality of the angles under the base, he is enabled to demonstrate that the triangle is isosceles? For if every right line, standing upon a right line, and forming two angles, makes them equal to two right; when the angles under the base are equal, those upon the base will be equal. And these being equal, the sides subtending them shall be equal. Euclid, therefore, having used this in the whole elementary institution, was enabled to conclude, that when the angles under the base are equal, the triangle is isosceles. Indeed he requires this also, for the demonstration of certain theorems: For shortly a theorem will appear, evincing, that if a right line standing on a right line, forms angles, it will either make two right, or angles equal to two right. And the theorems, indeed, preceding this, require no such conversion; but those which follow, are indigent of this, and establish their credibility from the present theorem.