The Commentaries of Proclus on the First Book of Euclid's Elements of Geometry Translated by Thomas Taylor (London, 1792) Proposition 4

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[Thomas Taylor, *The Philosophical and Mathematical Commentaries of Proclus*, Vol. 2, pp. 41–49 (1792).]

PROPOSITION IV. Theorem I.

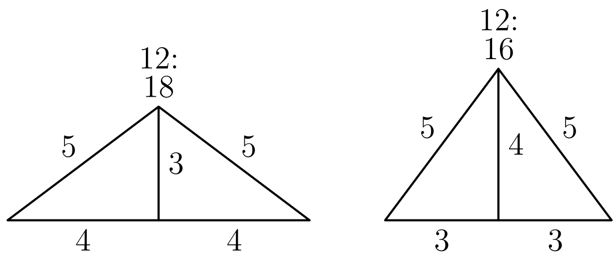
> If two triangles have two sides equal each to each; and have likewise the angles equal; which are comprehended by the equal sides; then they shall have their bases equal; and the two triangles shall be equal; and the remaining angles opposite to the equal sides shall be equal.

This is the first theorem in the institution of the elements, for all those which preceded were problems. The first, indeed, treating concerning the origin of triangles: but the second and third proposing to procure one right line equal to another. And of these the one produced an equal from an unequal line, but the other discovered an equal line by an ablation from one unequal. Since, therefore, equality, which is the first symptom in quantity, is to be constructed by us in a triangle and right line, it is delivered in the following theorem. For how can he who has not previously constructed triangles, and procured their origin, be learned in their essential accidents, and in the equality of angles and sides which they contain? How can he receive sides equal to sides, and right lines to other right lines, who has neither problematically investigated these, nor fabricated the invention of equal right lines? For if he should say it may happen before they are fabricated, that if two triangles have *this* for a *symptom*, they shall likewise have *this particular symptom*; would it not, in this case, be easy to object to him, that we by no means know whether a triangle can be constructed? And should it be afterwards inferred, that if there are two triangles, they may have two sides equal to two sides, may we not also doubt this, whether it is possible that right lines may be mutually equal? And this particularly in geometrical forms, in which inequality not entirely existing, equality is likewise inherent. For we must learn that the cornicular is always unequal to an acute angle, and the same is true of the semicircular angle, and the transition from the greater to the less does not entirely take place through that which is equal. The institutor of the elements, therefore, first of all removing these objections, delivers also the construction of triangles (for it is common to three forms) and the origin of equal right lines, in a two-fold order. For he produces the one, not yet existing: but he acquires the other by an ablation from an unequal line. But after these he very properly subjoins the theorem, by which it is shewn how triangles having two sides equal to two, each to each, and the angles comprehended by the equal sides equal, have also the base equal to the base, the area equal to the area, and the remaining angles to the remaining angles. For there are three particulars exhibited in these triangles: but two data. Hence, the equality of the two sides is given, or two equal sides (and it is manifestly given in proportion) and the equality of the angle contained by the equal sides: but three particulars are investigated, the equality of *base* to *base*, of *triangle* to *triangle*, and of the *remaining angles*. But because it is possible that triangles may have two sides equal to two, and yet the theorem not be true, because the one is not equal to the other, but both together, on this account he adds in the data, that the sides are equal not simply, but one to the other. For if one of the triangles should have one of its sides of three units, but the other of four; and again, if the sides of the other triangle are respectively two, and five units, the angle comprehended by these being right, the two sides of the one triangle will, indeed, taken together, be equal to the two sides of the other, or to seven units, yet the two triangles will not be equal. For the area of the one is six units[135], but of the other five. And the reason of this is, because the sides are not equal each to each. Hence, many, not observing this in the division of land, when they have received a greater, have thought it just the same as if they had received an equal field; and this because both the sides containing one field, have been together equal to both the sides containing the other field. It is requisite, therefore, to receive the one equal to the other, and to mark wherever the institutor of the elements subjoins this, because he does not add it without occasion. For discoursing on the equality of equal angles, he adds the particle *comprehended by equal sides*, lest by speaking indeterminately we should assume some one of the angles at the bases. Besides, when in triangles no side is previously named, we must conceive the base to be the side opposite to our sight; but when two are previously received, the remaining side is necessarily the base. Hence, here too, the institutor of the elements, having previously assumed two sides equal to two, calls the remainder the bases of the triangles. But a triangle is then said to be equal to a triangle, when their areas are equal. for it is possible, that though the ambits are equal, yet the areas may be unequal, on account of the inequality of angles. But I call the area, the space intercepted by the sides of the triangle: as also I denominate the ambit, the line composed from the three triangular sides. Each, therefore, is different, and it is requisite, indeed,

[135]This is easily proved from the mensuration of a triangular space, which it is well known is obtained by multiplying the base into half the altitude; and this in the first triangle will be equal to 3 multiplied by 2; and in the second, to 2 multiplied by $2\frac{1}{2} = 5$.

that besides the equality of the ambits, according to each side, the angles should also be equal, if also area ought to be equal to area. But it happens in certain triangles, that though the areas are equal, yet the ambits are unequal; and that the ambits being equal, the areas are unequal. For if there be two isosceles triangles, each of whose equal sides contains five units, but the base of the one is eight, and of the other six unit; he who is ignorant of geometry, will say that the greater triangle is that whose base contains eight units. For the whole ambit will be eighteen. But the geometrician will say, that the area of each triangle contains twelve units, and this he will demonstrate, by drawing in each triangle a perpendicular from the vertex, and multiplying this with either part of the segments of the base[136]. Butf it happens (as I have said) that though the ambits are equal, the spaces are unequal. Hence certain persons formerly fraudulently deceived their partners in the division of fields, on account of the equality according to ambit, receiving a larger field. But one base is said to be equal to another, and one right line to another, when their extremes conjoined make the whole coincide with the whole. For every right line, indeed, agrees with every right line; but equal right lines mutually coincide according to their extremes. Again, one right-lined angle is said to be equal to another, when one of the comprehending sides of one angle being placed upon one of the other, the remaining side also coincides with the remainder: but when one of the remaining sides falls external to the other, the greater angle is that whose side falls externally; and the less whose side falls within. For there, indeed, the one contains, but in this case it is contained. But we must assume the equality of angles according to the convenience of sides in right lines, and in all of the same species, as in lunulars and systroides[137], and figures on both sides convex;

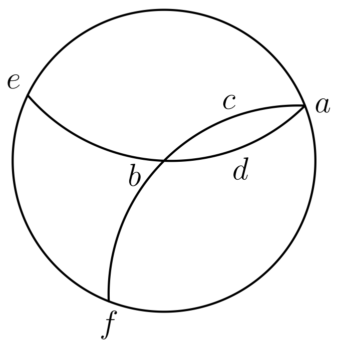
[136]The quantity of this perpendicular in each triangle may be easily obtained from the 47th proposition of this book; for in the first triangle it will be three units; and in the second four. Hence the area of each will be 12 units; but the ambit of the one will be 18, and of the other 16 units, as is evident in the following figures.



[137]That is angles formed from the circumferences of circles cutting or touching each other, when they are on both sides concave

because, it is possible that they may be equal, and yet the sides not mutually coincide. For a right angle is equal to a certain lunular angle, and yet it is not possible that right lines can coincide with circumferences. Besides, this also must be previously understood, that the angles are said to subtend the opposite sides. For every triangular angle is contained by two sides of the triangle, but is subtended by the remaining side. Hence, the geometrician, when he says that the angles are equal, adds, *which are opposite to the equal sides*, lest we should conceive it of no consequence whatever angle is received, and should think that he denominated any other two angles of the triangles equal, but we must call those equal which subtend equal sides. For equal sides mutually subtend equal angles. And such are the considerations necessary to the declaration of the present theorem.

But against the objection of our adversary[138], this must be previously assumed, that two right lines cannot comprehend space. For this the geometrician receives as evident. For if (says he) the extremes of the bases mutually coincide, the bases also shall coincide: but if not two right lines, will comprehend space. From whence, therefore, is the impossibility of this derived? Let there then be two right lines comprehending space $a\,c\,b$, $a\,d\,b$, and let them be infinitely produced. Then with the centre b, and interval $a\,b$, let a circle $a\,e\,f$ be described. Because, therefore, the line $a\,c\,b\,f$ is a



diameter, $a\,e\,f$ is the half of the circumference. Again, because $a\,d\,b\,e$ is a diameter, $a\,e$, likewise, is one half of the circumference. Hence $a\,e$, and $a\,e\,f$ are equal to the circumference, which is impossible. Two right lines therefore, cannot comprehend space; which the institutor of the elements knowing said, in the first Petition, *from every point, to every point, to draw a right line*, because one right line is always capable of uniting two points, but this is impossible for two right lines to effect. Many circumferences, indeed, may conjoin two points, both in the same, and in contrary parts: for by this means the extremities of a diameter conjoin two circumferences, but only

[138]Most probably Zeno, the Epicurean.

one right line. But it is possible that both within and without semicircles, infinite circumferences conjoining given points may be described. And the reason of this is, because a right line is the least of lines, having the same extremes. But there is every where one *minimum*, and this always becomes the measure of the infinity of others. As therefore a right line, since it is one, becomes the measure of the infinity of right-lined angles (for by this we discover their quantity) so likewise a right line procures us the greatest utility in the mensuration of such as are non-rectilineal. And thus much may suffice concerning these.

But that the whole demonstration of the present theorem depends on common conceptions, rising as it were spontaneously, and emerging from the evidence of hypotheses, is manifest to every one. For since two sides are equal to two sides, each to each, they will mutually coincide. But since the angles contained by the equal sides are equal, they also shall mutually coincide. And when angle is placed on angle, and sides on sides, so as to touch, in every part, the extremities of the sides beneath shall also coincide. But if these, then *base*, shall agree with *base*. And if three with three, the whole triangle shall accord with the whole triangle, and all shall be equal to all. Hence, therefore, equality considered in things of the same species, appears to be the cause of the whole demonstration. For here are two axioms endued with a power of containing the whole method of the proposed theorem. One, indeed, affirming, that *things which mutually coincide, are equal*; and this is simply true, requiring no limitation, and is employed by the institutor of the elements both in the base, and in the space, and in the other angles. For these, says he, are equal, because they mutually coincide. But the other affirming that *things which are equal mutually coincide.* This, however, is not true in all, but in those of a similar species. But I call things similar in species, such as a right line when compared with a right line, one circumference with another of the same circle, and the angles comprehended by similar lines endued with a similar position. But of these, I say, that such as are equal, mutually coincide: so that in short, the whole demonstration is of this kind. These equals, therefore are given, viz. two sides equal to two sides,and the angles which they comprehend, and these accord among themselves. But if these mutually coincide, the base also shall agree with the base, and all coincide with all. And if these accord, they are also equal. If then these are equal, it may at the same time be shewn that all are equal to all. And this appears to be the first mode of knowing triangles on all sides equal. And thus much concerning the whole demonstration.

But Carpus, the mechanist, who, in an astrological treatise, discourses of problems and theorems, says, "that they must not be passed over in silence, since they opportunely present themselves for investigation;" and lastly, entering on their distinction, he observes, "that the problematical genus precedes theorems in order. For in problems (says he) the invention of subjects is investigated prior to symptoms. Likewise a problematical proposition is simple, and requires no artificial intelligence. For this commands us to accomplish something evident, as *to construct an equilateral triangle*, or *from two given unequal right lines, to cut off from the greater a part equal to the less*. For what is there in these difficult and obscure. But he affirms that the proposition of a theorem is difficult, and requires the most accurate power, and a judgment productive of science, that it may appear neither to exceed, nor to be deficient from truth; such, indeed, as the present, which is the first of theorems. Add to, that in problems, there is one common way invented by resolution, by proceeding according to which, we can happily accomplish our purpose. For after this manner the more easy kind of problems are investigated. But the treatise of theorems is so very difficult, that even to our time (says he) no one has been able to deliver any common method of their invention. Hence, on account of facility also, the problematic genus is more simple. But these being distinguished, it is on this account (says he) that in the elementary institution problems precede theorems, and from these the institution of the elements begins; and the first theorem is in order the fourth, not because the fourth is exhibited from the preceding, but because it is necessary they should precede as being problems, and this a theorem, though it should require none of the antecedent propositions for its demonstration. For the present theorem entirely employs common conceptions; and in a certain respect receives the same triangle in a different position. Since coincidence, and its consequent equality possess a sensible and manifest apprehension. But such being the demonstration of the first theorem, problems with great propriety precede, because they are universally allotted the primary place." And perhaps, indeed, problems antecede theorems in order; and particularly among those who ascend to contemplation from the arts, which are conversant with sensible particulars; but theorems excel problems in dignity of nature. And it appears, that all geometry, so far as it conjoins itself with a variety of arts, energizes problematically: but so far as it coheres to the first science, it proceeds theorematically from problems to theorems, from things secondary to such as are first, and from things which more regard the arts, to such as are endued with a greater power of producing science. It is, therefore, vain to accuse Geminus, for affirming that theorems are prior to problems. For Carpus assigns a precedency to problems, according to order: but Geminus to theorems, according to a more perfect dignity. But of this fourth theorem, we have already observed, that in a certain respect it is indigent of the preceding problems, in which we learn the origin of triangles, and the invention of equality. But we now add, that since it is the most simple and principle [*sic.*] of theorems (for it is naturally, as I may say, exhibited from primary conceptions alone), but demonstrates a certain symptom appearing about triangles, having two sides equal to two, each to each, and the two angles equal contained by the equal sides, it is with great propriety placed the first after problems, in which things subject to this symptom, and the data themselves are constructed.