[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 350–368 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 47.]

1. **the square on**, τὸ ἀπὸ τετράγωνον, the word ἀναγραφέν or ἀναγεγραμμένον being understood.

   **subtending the right angle**. Here ὑποτεινούσης, "subtending," is used with the simple accusative (τὴν ὀρθὴν γωνίαν) instead of being followed by ὑπό and the accusative, which seems to be the original and more orthodox construction. Cf. I. 18, note.

33. **the two sides AB, BD . . .** . Euclid actually writes "$DB$, $BA$," and therefore the equal sides in the two triangles are not mentioned in corresponding order, though he adheres to the words ἑκατέρα ἑκατέρα "respectively." Here $DB$ is equal to $BC$ and $BA$ to $FB$.

44. [**But the doubles of equals are equal to one another.**] Heiberg brackets these words as an interpolation, since it quotes a *Common Notion* which is itself interpolated. Cf. notes on I. 37, p. 332, and on interpolated *Common Notions*, pp. 223–4.

"If we listen," says Proclus (p. 426, 6 sqq.), to those who wish to recount ancient history, we may find some of them referring this theorem to Pythagoras and saying that he sacrificed an ox in honour of his discovery. But for my part, while I admire those who first observed the truth of this theorem, I marvel more at the writer of the Elements, not only because he made it fast (κατεδήσατο) by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irrefragable arguments of science in the sixth Book. For in that Book he proves generally that, in right-angled triangles, the figure on the side subtending the right angle is equal to the similar and similarly situated figures described on the sides about the right angle."

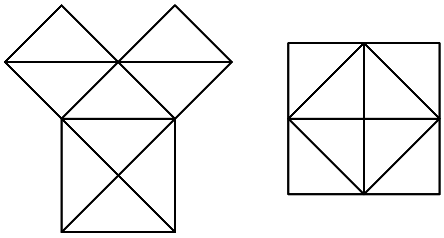
In addition, Plutarch (in the passages quoted above in the note on I. 44), Diogenes Laertius (VIII. 12) and Athenaeus (X. 13) agree in attributing this proposition to Pythagoras. It is easy to point out, as does G. Junge ("Wann haben die Griechen das Irrationale entdeckt?" in *Novae Symbolae Joachimicae*, Halle a. S., 1907, pp. 221–264), that these are late witnesses, and that the Greek literature which we possess belonging to the first five centuries after Pythagoras contains no statement specifying this or any other particular great geometrical discovery as due to him. Yet the distich of Apollodorus the "calculator," whose date (though it cannot be fixed) is at least earlier than that of Plutarch and presumably of Cicero, is quite definite as to the existence of *one* "famous proposition" discovered by Pythagoras, whatever it was. Nor does Cicero, in commenting apparently on the verses (*De nat. deor.* III. c. 36, §88), seem to dispute the fact of the geometrical discovery, but only the story of the sacrifice. Junge naturally emphasises the apparent uncertainty in the statements of Plutarch and Proclus. But, as I read the passages of Plutarch, I see nothing in them inconsistent with the supposition that Plutarch unhesitatingly accepted as discoveries of Pythagoras *both* the theorem of the square of the hypotenuse and the problem of the application of an area, and the only doubt he felt was as to which of the two discoveries was the more appropriate occasion for the supposed sacrifice. There is also other evidence not without bearing on the question. The theorem is closely connected with the whole of the matter of Eucl. Book II., in which one of the most prominent features is the use of the *gnomon*. Now the gnomon was a well-understood term with the Pythagoreans (cf. the fragment of Philolaus quoted on p. 141 of Boeckh's *Philolaos des Pythagoreers Lehren*, 1819). Aristotle also (*Physics* III. 4, 203 a 10–15) clearly attributes to the Pythagoreans the placing of odd numbers as *gnomons* round successive squares beginning with I, thereby forming new squares, while in another place (*Categ.* 14, 15 a 30) the word *gnomon* occurs in the same (obviously familiar) sense: "e.g., a square, when a *gnomon* is placed round it, is increased in size but is not altered in form." The inference must therefore be that practically the whole doctrine of Book II. is Pythagorean. Again Heron (? 3rd cent. A.D.), like Proclus, credits Pythagoras with a general rule for forming right-angled triangles with rational whole numbers for sides. Lastly the "summary" of Proclus appears to credit Pythagoras with the discovery of the theory, or study, of irrationals (τὴν τῶν ἀλόγων πραγματείαν). But it is now more or less agreed that the reading here should be, not τῶν ἀλόγων, but τῶν ἀναλόγων, or rather τῶν ἀνὰ λόγον ("of proportionals"), and that the author intended to attribute to Pythagoras a theory of *proportion*, i.e. the (arithmetical) theory of proportion applicable only to commensurable magnitudes, as distinct from the theory of Eucl. Book V., which was due to Eudoxus. It is not however disputed that the *Pythagoreans* discovered the irrational (cf. the scholium No. 1 to Book X.). Now everything goes to show that this discovery of the irrational was made with reference to $\sqrt{2}$, the ratio of the diagonal of a square to its side. It is clear that this presupposes the knowledge that I. 47 is true of an isosceles right-angled triangle; and the fact that some triangles of which it had been discovered to be true were *rational* right-angled triangles was doubtless what suggested the inquiry whether the ratio between the lengths of the diagonal and the side of a square could also be expressed in whole numbers. On the whole, therefore, I see not sufficient reason to question the tradition that, *so far as Greek geometry is concerned* (the possible priority of the discovery of the same proposition in India will be considered later), Pythagoras was the first to introduce the theorem of I. 47 and to give a general proof of it.

On this assumption, how was Pythagoras led to this discovery? It has been suggested and commonly assumed that the Egyptians were aware that a triangle with its sides in the ratio 3, 4, 5 was right-angled. Cantor inferred this from the fact that this was precisely the triangle with which Pythagoras began, if we may accept the testimony of Vitruvius (IX. 2) that Pythagoras taught how to make a right angle by means of three lengths measured by the numbers 3, 4, 5. If then he took from the Egyptians the triangle 3, 4, 5, he presumably learnt its property from them also. Now the Egyptians must certainly be credited from a period at least as far back as 2000 B.C. with the knowledge that $4^2 + 3^2 = 5^2$. Cantor finds proof of this in a fragment of papyrus belonging to the time of the 12th Dynasty newly discovered at Kahun. In this papyrus we have extractions of square roots: e.g. that of 16 is 4, that of $1\frac{9}{16}$ is $1\frac{1}{4}$, that of $6\frac{1}{4}$ is $2\frac{1}{2}$, and the following equations can be traced:

$$\begin{aligned} 1^2 + \left(\tfrac{3}{4}\right)^2 &= \left(1\tfrac{1}{4}\right)^2 \\ 8^2 + 6^2 &= 10^2 \\ 2^2 + \left(1\tfrac{1}{2}\right)^2 &= \left(2\tfrac{1}{2}\right)^2 \\ 16^2 + 12^2 &= 20^2. \end{aligned}$$

It will be seen that $4^2 + 3^2 = 5^2$ can be derived from each of these by multiplying, or dividing out, by one and the same factor. We may therefore admit that the Egyptians knew that $3^2 + 4^2 = 5^2$. But there seems to be no evidence that they knew that the triangle $(3, 4, 5)$ is *right-angled*; indeed, according to the latest authority (T. Eric Peet, *The Rhind Mathematical Papyrus*, 1923), nothing in Egyptian mathematics suggests that the Egyptians were acquainted with this or any special cases of the Pythagorean theorem.

How then did Pythagoras discover the general theorem? Observing that 3, 4, 5 was a right-angled triangle, while $3^2 + 4^2 = 5^2$, he was probably led to consider whether a similar relation was true of the sides of right-angled triangles other than the particular one. The simplest case (geometrically) to investigate was that of the *isosceles* right-angled triangle; and the truth of the theorem in this particular case would easily appear from the mere construction of a figure. Cantor ($\text{I}_3$, p. 185) and Allman (*Greek Geometry from Thales to Euclid*, p. 29) illustrate by a figure in which the squares are drawn outwards, as in I. 47, and divided by diagonals into equal triangles; but I think that the truth was more likely to be first observed from a figure of the kind suggested by Bürk (*Das Āpastamba-Śulba-Sūtra* in *Zeitschrift der deutschen morgenländ. Gesellschaft*, LV., 1901, p. 557) to explain how



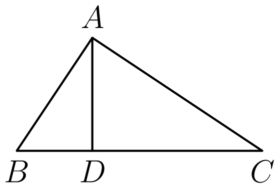
the Indians arrived at the same thing. The two figures are as shown above. When the geometrical consideration of the figure had shown that the isosceles right-angled triangle had the property in question, the investigation of the same fact from the arithmetical point of view would ultimately lead to the other momentous discovery of the irrationality of the length of the diagonal or a square expressed in terms of its side.

The *irrational* will come up for discussion later; and our next question is: Assuming that Pythagoras had observed the geometrical truth of the theorem in the case of the two particular triangles, and doubtless of other rational right-angled triangles, how did he establish it generally?

There is no positive evidence on this point. Two possible lines are however marked out. (1) Tannery says (*La Géométrie grecque*, p. 105) that the geometry of Pythagoras was sufficiently advanced to make it possible for him to prove the theorem by *similar triangles*. He does not say in what particular manner similar triangles would be used, but their use must apparently have involved the use of *proportions*, and, in order that the proof should be conclusive, of the theory of proportions in its complete form applicable to incommensurable as well as commensurable magnitudes. Now Eudoxus was the first to make the theory of proportion independent of the hypothesis of commensurability; and as, before Eudoxus' time, this had not been done, any proof of the general theorem by means of proportions given by Pythagoras must at least have been inconclusive. But this does not constitute any objection to the supposition that the truth of the general theorem may have been discovered in such a manner; on the contrary, the supposition that Pythagoras proved it by means of an imperfect theory of proportions would better than anything else account for the fact that Euclid had to devise an entirely new proof, as Proclus says he did in I. 47. This proof had to be independent of the theory of proportion even in its rigorous form, because the plan of the *Elements* postponed that theory to Books V. and VI., while the Pythagorean theorem was required as early as Book II. On the other hand, if the Pythagorean proof had been based on the doctrine of Books I. and II. only, it would scarcely have been necessary for Euclid to supply a new proof.

The possible proofs by means of proportion would seem to be practically limited to two.

(a) One method is to prove, from the similarity of the triangles $ABC$, $DBA$, that the rectangle $CB$, $BD$ is equal to the square on $BA$, and, from the similarity of the triangles $ABC$, $DAC$, that the rectangle $BC$, $CD$ is equal to the square on $CA$; whence the result follows by addition.

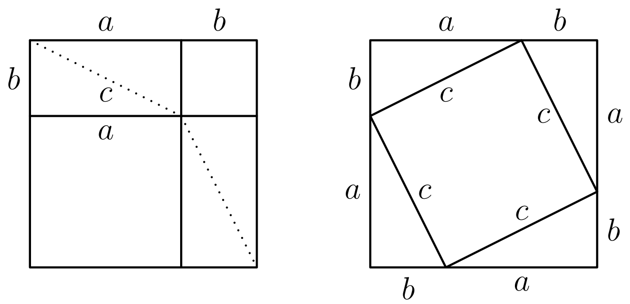



It will be observed that this proof is *in substance* identical with that of Euclid, the only difference being that the equality of the two smaller squares to the respective rectangles is inferred by the method of Book VI. instead of from the relation between the areas of parallelograms and triangles on the same base and between the same parallels established in Book I. It occurred to me whether, if Pythagoras' proof had come, even in substance, so near to Euclid's, Proclus would have emphasised so much as he does the originality of Euclid's, or would have gone so far as to say that he marvelled more at that proof than at the original discovery of the theorem. But on the whole I see no difficulty; for there can be little doubt that the proof by proportion is what suggested to Euclid the method of I. 47, and the transformation of the method of proportions into one based on Book I. only, effected by a construction and proof so extraordinarily ingenious, is a veritable *tour de force* which compels admiration, notwithstanding the ignorant strictures of Schopenhauer, who wanted something as obvious as the second figure in the case of the isosceles right-angled triangle (p. 352), and accordingly (*Sämmtliche Werke*, III. § 39 and I. § 15) calls Euclid's proof "a mouse-trap proof" and "a proof walking on stilts, nay, a mean, underhand, proof" ("Des Eukleides stelzbeiniger, ja, hinterlistiger Beweis").

(b) The other possible method is this. As it would be seen that the triangles into which the original triangle is divided by the perpendicular from the rightangle on the hypotenuse are similar to one another and to the whole triangle, while in these three triangles the two sides about the right angle in the original triangle, and the hypotenuse of the original triangle, are corresponding sides, and that the sum of the two former similar triangles is identically equal to the similar triangle on the hypotenuse, it might be inferred that the same would also be true of *squares* described on the corresponding three sides respectively, because squares as well as similar triangles are to one another in the duplicate ratio of corresponding sides. But the same thing is equally true of any similar rectilinear figures, so this proof would practically establish the extended theorem of Eucl. VI. 31, which theorem, however, Proclus appears to regard as being entirely Euclid's discovery.

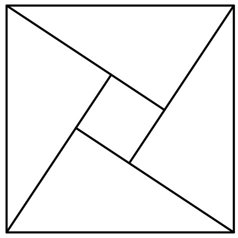
On the whole, the most probable supposition seems to me to be that Pythagoras used the first method (a) of proof by means of the theory of proportion as he knew it, i.e. in the defective form which was in use up to the date of Eudoxus.

(2) I have pointed out the difficulty in the way of the supposition that Pythagoras' proof depended upon the principles of Eucl. Books I. and II. only.

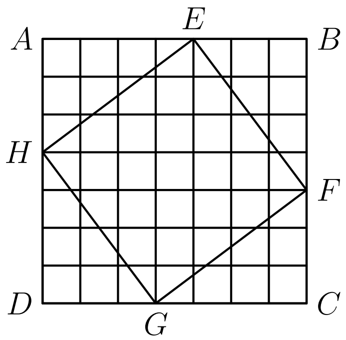



Were it not for this difficulty, the conjecture of Bretschneider (p. 82), followed by Hankel (p. 98), would be the most tempting hypothesis. According to this suggestion, we are to suppose a figure like that of Eucl. II. 4 in which $a$, $b$ are the sides of the two inner squares respectively, and $a + b$ is the side of the complete square. Then, if the two complements, which are equal, are divided by their two diagonals into four equal triangles of sides $a$, $b$, $c$, we can place these triangles round another square of the same size as the whole square, in the manner shown in the second figure, so that the sides $a$, $b$ of *successive* triangles make up one of the sides of the square and are arranged in cyclic order. It readily follows that the remainder of the square when the four triangles are deducted is, in the one case, a square whose side is $c$, and in the other the sum of two squares whose sides are $a$, $b$ respectively. Therefore the square on $c$ is equal to the sum of the squares on $a$, $b$. All that can be said against this conjectural proof is that it has no specifically Greek colouring but rather recalls the Indian method. Thus Bhāskara (born 1114 A.D.; see Cantor, I$_3$, p. 656) simply draws four right-angled triangles equal to the original one inwards, one on each side of the square on the hypotenuse, and says "see!", without even adding that inspection shows that

$$c^2 = 4\,\frac{ab}{2} + (a-b)^2 = a^2 + b^2.$$



Though, for the reasons given, there is difficulty in supposing that Pythagoras used a general proof of this kind, which applies of course to right-angled triangles with sides incommensurable as well as commensurable, there is no objection, I think, to supposing that the truth of the proposition in the case of the first *rational* right-angled triangles discovered, e.g., 3, 4, 5, was proved by a method of this sort. Where the sides are commensurable in this way, the squares can be divided up into small (unit) squares, which would much facilitate the comparison between them. That this subdivision was in fact resorted to in adding and subtracting squares is made probable by Aristotle's allusion to odd numbers as *gnomons* placed round unity to form successive squares in *Physics* III. 4; this must mean that the squares were represented by dots arranged in the form of a square and a gnomon formed of dots put round, or that (if the given square was drawn in the usual way) the gnomon was divided up into unit squares. Zeuthen has shown ("*Théorème de Pythagore," Origine de la Géometrie scientifique* in *Comptes rendus du II^me^ Congrès international de Philosophie*, Genève, 1904), how easily the proposition could be proved by a method of this kind for the triangle 3, 4, 5. to admit of the two smaller squares being shown side by side, take a square on a line containing 7 units of length $(4 + 3)$, and divide it up into 49 small squares. It would be obvious that the whole square could be exhibited as containing four rectangles of sides 4, 3 cyclically arranged round the figure with one unit square in the middle. (This same figure is given by Cantor, $\mathrm{I}_3$, p. 680. to illustrate the method given in the Chinese "Chóu-peï".) It would be seen that



(i) the whole square ($7^2$) is made up of two squares $3^2$ and $4^2$, and two rectangles 3, 4;

(ii) the same square is made up of the square $EFGH$ and the halves of four of the same rectangles 3, 4, whence the square $EFGH$, being equal to the sum of the square $3^2$ and $4^2$, must contain 25 unit squares and its side, or the diagonal of one of the rectangles, must contain 5 units of length.

Or the result might equally be seen by observing that

(i) the square $EFGH$ on the diagonal of one of the rectangles is made up of the halves of four rectangles and the unit square in the middle, while

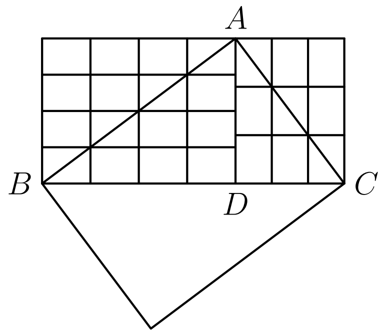
(ii) the squares $3^2$ and $4^2$ placed at adjacent corners of the large square make up two rectangles 3, 4 with the unit square in the middle.

The procedure would be equally easy for any *rational* right-angled triangle, and would be a natural method for trying to *prove* the property when it had once been *empirically* observed that triangles like 3, 4, 5 did in fact contain a right angle.

Zeuthen has, in the same paper, shown in a most ingenious way how the property of the triangle 3, 4, 5 could be verified by a sort of combination of the second possible method by similar triangles, (b) on p. 354 above, with subdivision of *rectangles* into similar small *rectangles.* I give the method on account of its interest, although it is no doubt too advanced to have been used by those who first proved the property of the particular triangle.

Let $ABC$ be a triangle right-angled at $A$, and such that the lengths of the sides $AB$, $AC$ are 4 and 3 units respectively.

Draw the perpendicular $AD$, divide up $AB$, $AC$ into unit lengths, complete the triangle on $BC$ as base and with $AD$ as altitude, and subdivide this rectangle into small rectangles by drawing parallels to $BC$, $AD$ through the points of division of $AB$, $AC$.




Now, since the diagonals of the small rectangles are all equal, each being of unit length, it follows by similar triangles that the small rectangles are all equal. And the rectangle with $AB$ for diagonal contains 16 of the small rectangles, while the rectangle with diagonal $AC$ contains 9 of them.

But the sum of the triangles $ABD$, $ADC$ is equal to the triangle $ABC$.

Hence the rectangle with $BC$ as diagonal contains $9 + 16$ or 25 of the small rectangles;
and therefore $BC = 5$.

**Rational right-angled triangles from the arithmetical standpoint.**

Pythagoras investigated the *arithmetical* problem of finding rational numbers which could be made the sides of right-angled triangles, or of finding square numbers which are are sum of two squares; and herein we find the beginning of the *indeterminate analysis* which reached so high a stage of development in Diophantus. Fortunately Proclus has preserved Pythagoras' method of solution in the following passage (pp. 428, 7—429, 8). "Certain methods for the discovery of triangles of this kind are handed down, one of which they refer to Plato, and another to Pythagoras. [The latter] starts from odd numbers. For it makes the odd number the smaller of the sides about the right angle; then it takes the square of it, subtracts unity, and makes half the difference the greater of the sides about the right angle; lastly it adds unity to this and so forms the remaining side, the hypotenuse. For example, taking 3, squaring it, and subtracting unity from the 9, the method takes half of the 8, namely 4; then, adding unity to it again, it makes 5, and a right-angled triangle has been found with one side 3, another 4 and another 5. But the method of Plato argues from even numbers. For its takes the given even number and makes it one of the sides about the right angle; then, bisecting this number and squaring the half, it adds unity to the square to form the hypotenuse, and subtracts unity from the square to form the other side about the right angle. For example, taking 4, the method squares half of this, or 2, and so makes 4; then subtracting unity, it produces 3, and adding unity it produces 5. Thus it has formed the same triangle as that which was obtained by the other method."

The formula of Pythagoras amounts, if $m$ be an odd number, to

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2,$$

the sides of the right-angled triangle being $m$, $\dfrac{m^2-1}{2}$, $\dfrac{m^2+1}{2}$. Cantor (I$_3$, pp. 185–6), taking up an idea of Röth (*Geschichte der abendländischen Philosophie*, II. 527), gives the following as a possible explanation of the way in which Pythagoras arrived at his formula. If $c^2 = a^2 + b^2$, it follows that

$$a^2 = c^2 - b^2 = (c+b)(c-b).$$

Numbers can be found satisfying the first equation if (1) $c+b$ and $c-b$ are either both even or both odd, and if further $c+b$ and $c-b$ are such even numbers as, when multiplied together, produce a square number. The first condition is necessary because, in order that $c$ and $b$ may both be whole numbers, the sum and difference of $c+b$ and $c-b$ must both be even. The second condition is satisfied if $c+b$ and $c-b$ are what were called *similar* numbers (ὅμοιοι ἀριθμοί); and that such numbers were most probably known in the time before Plato may be inferred from their appearing in Theon of Smyrna (*Expositio rerum mathematicarum ad legendum Platonem utilium*, ed. Hiller, p. 36, 12), who says that similar plane numbers are, first, all square numbers and, secondly, such oblong numbers as have the sides which contain them proportional. Thus 6 is an oblong number with length 3 and breadth 2; 24 is another with length 6 and breadth 4. Since therefore 6 is to 3 as 4 is to 2, the numbers 6 and 24 are similar.

Now the simplest case of two similar numbers is that of 1 and $a^2$, and, since 1 is odd, the condition (1) requires that $a^2$, and therefore $a$ is also odd. That is, we may take 1 and $(2n+1)^2$ and equated them respectively to $c-b$ and $c+b$, whence we have

$$\begin{aligned} b &= \frac{(2n+1)^2-1}{2}, \\ c &= \frac{(2n+1)^2-1}{2}+1, \\ \text{while} \qquad a &= 2n+1. \end{aligned}$$

As Cantor remarks, the form in which $c$ and $b$ appear correspond sufficiently closely to the description in the text of Proclus.

Another obvious possibility would be, instead of equating $c-b$ to unity, to put $c-b=2$, in which case the similar number $c+b$ must be equated to double of some square, i.e. to a number of the form $2n^2$, or to the half of an even square number, say $\frac{(2n)^2}{2}$. This would give

$$\begin{aligned} a &= 2n, \\ b &= n^2-1, \\ c &= n^2+1, \end{aligned}$$

which is Plato's solution, as given by Proclus.

The two solutions supplement each other. It is intersting to obsere that the method suggested by Röth and Cantor is very like that of Eucl. X. (Lemma 1 following Prop. 28). We shall come to this later, but it may be mentioned here that the problem is *to find two square numbers such that* *their sum is also a square*. Euclid there uses the property of II. 6 to the effect that, if $AB$ is bisected at $C$ and produced to $D$,

$$AD \,.\, DB + BC^2 = CD^2.$$

We may write this

$$uv = c^2 - b^2,$$

where

$$u = c + b, \quad v = c - b.$$

In order that $uv$ may be a square, Euclid points out that $u$ and $v$ must be similar numbers, and further that $u$ and $v$ must be either both odd or both even in order that $b$ may be a whole number. We may then put for the similar numbers, say, $\alpha\beta^2$ and $\alpha\gamma^2$, whence (if $\alpha\beta^2$, $\alpha\gamma^2$ are either both odd or both even) we obtain the solution

$$\alpha\beta^2 \,.\, \alpha\gamma^2 + \left(\frac{\alpha\beta^2 - \alpha\gamma^2}{2}\right)^2 = \left(\frac{\alpha\beta^2 + \alpha\gamma^2}{2}\right)^2.$$

But I think a serious, and even fatal, objection to the conjecture of Cantor and Röth is the very fact that the method enables both the Pythagorean and the Platonic series of triangles to be deduced with equal ease. If this had been the case with the method used by Pythagoras, it would not, I think, have been left to Plato to discover the second series of such triangles. It seems to me therefore that Pythagoras must have used some method which would produce his rule *only*; and further it would be some less recondite method, suggested by direct *observation* rather than by argument from general principles.

One solution satisfying these conditions is that of Bretschneider (p. 83), who suggests the following simple method. Pythagoras was certainly aware that the successive odd numbers are *gnomons*, or the differences between successive square numbers. It was then a simple matter to write down in three rows ($a$) the natural numbers, ($b$) their squares, ($c$) the successive odd numbers consituting the differences between the successive squares in ($b$), thus:

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| | 1 | 4 | 9 | 16 | 25 | 36 | 49 | 64 | 81 | 100 | 121 | 144 | 169 | 196 |
| 1 | 3 | 5 | 7 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 | |

Pythagoras had then only to pick out the numbers in the third row which are squares, and his rule would be obtained by finding the formula connecting the square in the third line with the two adjacent squares in the second line. But even this would require some little argument; and I think a still better suggestion, because making pure observation play a greater part, is that of P. Treutlein (*Zeitschrift für Mathematik und Physik*, XXVIII., 1883, Hist.-litt. Abtheilung, pp. 209 sqq.).

We have the best evidence (e.g. in Theon of Smyrna) of the practice of representing square numbers and other figured numbers, e.g., oblong triangular, hexagonal, by dots or signs arranged in the shape of the particular figure. (Cf. Aristotle, *Metaph.* 1092 b 12). Thus, says, Treutlein, it would be easily seen that any square number can be turned into the next higher square by putting a single row of dots round two adjacent sides, in the form of a gnomon (see figures on next page).

If $a$ is the side of a particular square, the gnomon round it is shown by simple inspection to contain $2a+1$ dots or units. Now, in order that $2a+1$ may itself be a square, let us suppose

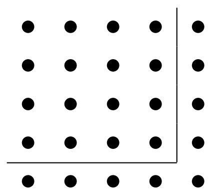
$$\begin{array}{lrcl} & 2a+1 & = & n^2 \\ \text{whence} & a & = & \tfrac{1}{2}(n^2-1), \\ \text{and} & a+1 & = & \tfrac{1}{2}(n^2+1). \end{array}$$

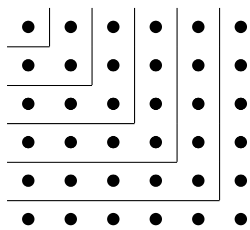
In order that $a$ and $a+1$ may be integral, $n$ must be odd, and we have at once the Pythagorean formula

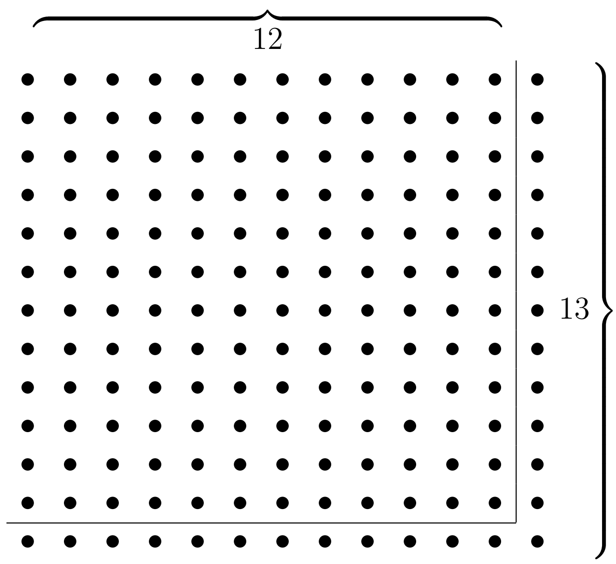
$$n^2 + \left(\frac{n^2-1}{2}\right)^2 = \left(\frac{n^2+1}{2}\right)^2.$$

I think Treutlein's hypothesis is shown to be the correct one by the passage in Aristotle's *Physics* already quoted, where the reference is undoubtedly to the Pythagoreans, and odd numbers are clearly identified with *gnomons* "placed round 1." But the ancient commentaries on the passage make the matter clearer still. Philoponus says: "As a proof ... the Pythagoreans refer to what happens with the addition of numbers; for when the odd numbers are successively added to a square number they keep it square and equilateral.... Odd numbers are accordingly called *gnomons* because, when added to what are already squares, they preserve the square form.... Alexander has excellently said in explanation that the phrase 'when gnomons are placed round' means *making a figure* with the odd numbers (τὴν κατὰ τοὺς περιττοὺς ἀριθμοὺς σχηματογραφίαν)... for it is the practice with the Pythagoreans to *represent things in figures* (σχηματογραφεῖν)."

The next question is: assuming this explanation of the Pythagorean formula, what are we to say of the origin of Plato's? It could of course be obtained as a particular case of the general formula of Eucl. X. already referred to; but there are two simple alternative explanations in this case also.



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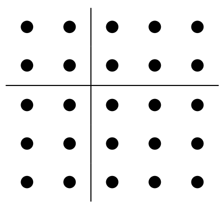
(1) Bretschneider observes that, to obtain Plato's formula, we have only to double the sides of the squares in the Pythagorean formula, for

$$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2,$$

where however $n$ is not necessarily odd.

(2) Treutlein would explain by means of an extension of the gnomon idea. As he says, the Pythagorean formula was obtained by placing a gnomon consisting of a single row of dots round two adjacent sides of a square, it would be natural to try whether another solution could not be found by placing round the square a *double* row of dots. Such a gnomon would equally turn the square into a larger square; and the question would be whether the double-row gnomon itself could be a square. If the side of the original square



was $a$, it would easily be seen that the number of units in the double-row gnomon would be $4a + 4$, and we have only to put

$$4a + 4 = 4n^2,$$

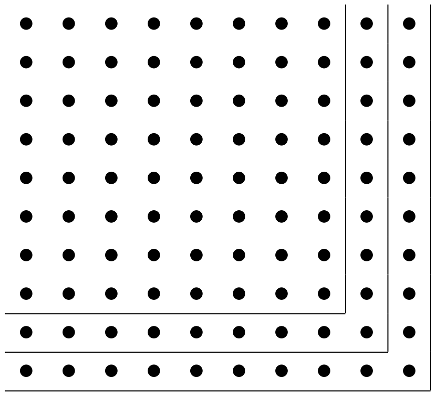
whence

$$\begin{aligned} a &= n^2 - 1, \\ a + 2 &= n^2 + 1, \end{aligned}$$

and we have the Platonic formula

$$(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2.$$

I think this is, in substance, the right explanation, but, in form, not quite correct. The Greeks would not, I think, have treated the *double* row as a gnomon. Their comparison would have been between (1) a certain square *plus* a single-row gnomon and (2) the same square *minus* a single row gnomon. As the application of Eucl. II. 4 to the case where the sides of the square are $a$, 1 enables the Pythagorean formula to be obtained as Treutlein obtains it, so I think that Eucl. II. 8 confirms the idea that the Platonic formula was



obtained by comparing a square *plus* a gnomon with the same square *minus* a gnomon. For II. 8 proves that

$$4ab + (a - b)^2 = (a + b)^2,$$

whence, substituting 1 for $b$, we have

$$4a + (a - 1)^2 = (a + 1)^2,$$

and we have only to put $a = n^2$ to obtain Plato's formula.

**The "theorem of Pythagoras" in India.**

This question has been discussed anew in the last few years as the result of the publication of two important papers by Albert Bürk on *Das Āpastama-Sulba-Sütra* in the *Zeitschrift der deutschen morgenländischen Gesellschaft* (LV., 1901, pp. 543–591, and LVI., 1902, pp. 327–391). The first of the two papers contains the introduction and the text, the second the translation with notes. A selection of the most important parts of the material was made and issued by G. Thibaut in the *Journal of the Asiatic Society of Bengal*, XLIV., 1875, Part 1. (reprinted also at Calcutta, 1875, as *The Śulvasūtras*, by G. Thibaut). Thibaut in this work gave a most valuable comparison of extracts from the three Śulvasūtras by Bāudhāyana, Āpastamba and Kātyāyana respectively, with a running commentary and an estimate of the date and originality of the geometry of the Indians. Bürk has however done good service by making the Āpastamba-Ś-S. accessible in its entirety and investigating the whole subject afresh. With the natural enthusiasm of an editor for the work he is editing, he roundly maintains, not only that the Pythagorean theorem was known and proved in all its generality by the Indians long before the date of Pythagoras (about 580–500 B.C.), but that they had also discovered the irrational; and further that, so far from Indian geometry being indebted to the Greek, the much-travelled Pythagoras probably obtained his theory from India (*loc. cit.* LV., p. 575 note). Three important notices and criticisms of Bürk's work have followed, by H. G. Zeuthen ("*Théorème de Pythagore,*" *Origine de la Géométrie scientifique*, 1904, already quoted), by Moritz Cantor (*Über die älteste indische Mathematik* in the *Archiv der Mathematik und Physik*, VIII., 1905, pp. 63–72) and by Heinrich Vogt (*Haben die alten Inder den Pythagoreischen Lehrsatz und das Irrationale gekannt?* in the *Bibliotheca Mathematica*, VII$_3$, 1906, pp. 6–23. See also Cantor's *Geschichte der Mathematik*, I$_3$, pp. 635–645.

The general effect of the criticisms is, I think, to show the necessity for the greatest caution, to say the least, in accepting Bürk's conclusions.

I proceed to give a short summary of the portions of the contents of the Āpastamba-Ś.-S. which are important in the present connexion. It may be premised that the general object of the book is to show how to construct altars of certain shapes, and to vary the dimensions of altars without altering the form. It is a collection of *rules* for carrying out certain constructions. There are no proofs, the nearest approach to a proof being in the rule for obtaining the area of an isosceles trapezium, which is done by drawing a perpendicular from one extremity of the smaller of the two parallel sides to the greater, and then taking away the triangle so cut off and placing it, the other side up, adjacent to the other equal side of the trapezium, thereby transforming the trapezium into a rectangle. It should also be observed that Āpastamba does not speak of *right-angled triangles*, but of two adjacent sides and the diagonal of a *rectangle*. For brevity, I shall use the expression "rational rectangle" to denote a rectangle the two sides and the diagonal of which can be expressed in terms of rational numbers. The references in brackets are to the chapters and numbers of Āpastamba's work.

(1) Constructions of right angles by means of cords of the following relative lengths respectively:

$$\left\{\begin{array}{rrrl} 3, & 4, & 5 & (\text{I. } 3, \text{V. } 3) \\ 12, & 16, & 20 & (\text{V. } 3) \\ 15, & 20, & 25 & (\text{V. } 3) \end{array}\right.$$

$$\left\{\begin{array}{rrrl} 5, & 12, & 13 & (\text{V. } 4) \\ 15, & 36, & 39 & (\text{I. } 2, \text{V. } 2, 4) \end{array}\right.$$

$$\begin{array}{rrrl} 8, & 15, & 17 & (\text{V. } 5) \end{array}$$

$$\begin{array}{rrrl} 12, & 35, & 37 & (\text{V. } 5) \end{array}$$

(2) A general enunciation of the Pythagorean theorem thus: "The diagonal of a rectangle produces [i.e. the square on the diagonal is equal to] the sum of what the longer and shorter sides separately produce [i.e. the squares on the two sides]." (I. 4)

(3) The application of the Pythagorean theorem to a *square* instead of a rectangle [i.e., to an *isosceles* right-angled triangle]: "The diagonal of a square produces an area double [of the original square]." (I. 5)

(4) An approximation to the value of $\sqrt{2}$; the diagonal of a square is $\left(1 + \frac{1}{3} + \frac{1}{3\,.\,4} - \frac{1}{3\,.\,4\,.\,34}\right)$ times the side. (I. 6)

(5) Application of this approximate value to the construction of a square with side of any length. (II. 1)

(6) The construction of $a\sqrt{3}$, by means of the Pythagorean theorem, as the diagonal of a rectangle with sides $a$ and $a\sqrt{2}$. (II. 2)

(7) Remarks equivalent to the following:

(a) $a\sqrt{\frac{1}{3}}$ is the side of $\frac{1}{9}(a\sqrt{3})^2$. or $a\sqrt{\frac{1}{3}} = \frac{1}{3}a\sqrt{3}$. (II. 3)

(b)

| | | | | | |
|---|---|---|---|---|---|
| A square on length of | 1 | unit gives | 1 | unit square | (III. 4) |
| " " | 2 | units gives | 4 | unit squares | (III. 6) |
| " " | 3 | " | 9 | " | (III. 6) |
| " " | $1\frac{1}{2}$ | " | $2\frac{1}{4}$ | " | (III. 8) |
| " " | $2\frac{1}{2}$ | " | $6\frac{1}{4}$ | " | (III. 8) |
| " " | $\frac{1}{2}$ | " | $\frac{1}{4}$ | " | (III. 10) |
| " " | $\frac{1}{3}$ | " | $\frac{1}{9}$ | " | (III. 10) |

(c) Generally, the square on any length contains as many rows (of small, unit, squares) as the length contains units. (III. 7)

(8) Constructions, by means of the Pythagorean theorem, of

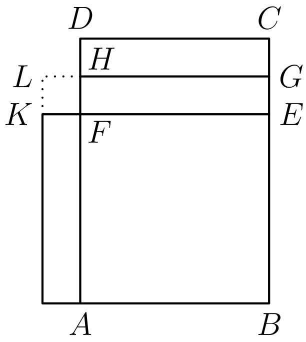
(a) the *sum* of two squares as one square, (II. 4)

(b) the *difference* of two squares as one square, (II. 5)

(9) A transformation of a rectangle into a square. (II. 7)

[This is not directly done as by Euclid in II. 14, but the rectangle is first transformed into a gnomon, i.e., into the difference between two squares, which difference is then transformed into one square by the preceding rule.

If $ABCD$ be the given rectangle of which $BC$ is the longer side, cut off the square $ABEF$, bisect the rectangle $DE$ left over by $HG$ parallel to $FE$, move the upper half $DG$ and place it on $AF$ as base in the position $AK$. Then the rectangle $ABCD$ is equal to the gnomon which is the difference




between the square $LB$ and the square $LF$. In other words, Āpastamba transforms the rectangle $ab$ into the difference between the squares $\left(\frac{a+b}{2}\right)^2$ and $\left(\frac{a-b}{2}\right)^2$.]

(10) An attempt at a transformation of a square ($a^2$) into a rectangle which shall have one side of a given length ($b$). (III. 1)

[This shows no sign of such a procedure as that of Eucl. I. 44, and indeed does no more than say that we must subtract $ab$ from $a^2$ and then adapt the remainder $a^2 - ab$ so that it may "fit on" to the rectangle $ab$. The problem is therefore only reduced to another of the same kind, and presumably it was only solved *arithmetically* in the case where $a$, $b$ are given numerically. The Indian was therefore far from the general, geometrical, solution.]

(11) Increase of a given square into a larger square. (III. 9)

[This amounts to saying that you must add two rectangles $(a, b)$ and anther square ($b^2$) in order to transform a square ($a^2$) into a square $(a + b)^2$. The formula is therefore that of Eucl. II. 4, $a^2 + 2ab + b^2 = (a + b)^2$.]

The first important question in relation to the above is that of date. Bürk assigns to the *Āpastamba-Śulva-Sūtra* a date at least as early as the 5th or 4th century B.C. He observes however (what is likely enough) that the matter of it must have been much older than the book itself. Further, as regards on of the constructions for right angles, that by means of cords of lengths 15, 36, 39, he shows that it was known at the time of the *Tāittirīya-Samhitā* and the *Satapatha-Brāhmana*, still older works belonging to the 8th century B.C. at latest. It may be that (as Bürk maintains) the discovery that triangles with sides $(a, b, c)$ in rational numbers such as $a^2 + b^2 = c^2$ are right-angled was nowhere made so early as in India. We find however in two ancient Chinese treatises (1) a statement that the diagonal of the rectangle $(3, 4)$ is 5 and (2) a rule for finding the hypotenuse of a "right triangle" from the sides, while tradition connects both works with the name of Chou Kung who died 1105 B.C. (D. E. Smith, *History of Mathematics*, I. pp. 30–33, II. p. 288).

As regards the various "rational rectangles" used by Apastamba, it is to be observed that two of the seven, viz. 8, 15, 17 and 12, 35, 37, do not belong to the Pythagorean series, the others consist of two which belong to it, viz. 3, 4, 5 and 5, 12, 13, and multiples of these. It is true, as remarked by Zeuthen (*op. cit.* p. 842), that the rules of II. 7 and III. 9 numbered (9) and (11) above respectively, would furnish the means of finding any number of "rational rectangles." But it would not appear that the Indians had been able to formulate any general rule: otherwise their list of such rectangles would hardly have been so meagre. Āpastamba mentions seven only, really reducible to four (though one other, 7, 24, 25, appears in the Bāudhāyana-Ś.-S., supposed to be older than Āpastamba). These are all that Āpastamba knew of, for he adds (V. 6): "So many *recognisable* (erkennbare) constructions are there," implying that he knew of no other "rational rectangles" that could be employed. But the words also imply that the theorem of the square on the diagonal is also true of other rectangles not of the "recognisable" kind, i.e., rectangles in which the sides and the diagonal are not in the ratio of integers; this is indeed implied by the constructions for $\sqrt{2}$, $\sqrt{3}$ etc. up to $\sqrt{6}$ (c.f. II. 2, VIII. 5). This is all that can be said. The theorem is, it is true, enunciated as a general proposition, but there is no sign of anything like a general proof; there is nothing to show that the assumption of its universal truth was founded on anything better than an imperfect induction from a certain number of cases, discovered empirically, of triangles with sides in the ratio of whole numbers in which the property (1) that the square on the longest side is equal to the sum of the squares on the other two sides was found to be always accompanied by the property (2) that the latter two sides include a right angle.

It remains to consider Bürk's claim that the Indians had discovered the *irrational*. This is based on the approximate value of $\sqrt{2}$ given by Āpastamba in his rule I. 6 numbered (4) above. There is nothing to show how this was arrived at, but Thibaut's suggestion certainly seems the best and most natural. The Indians may have observed that $17^2 = 289$ is nearly double of $12^2 = 144$. If so, the next question would would naturally occur to them would be, by how much the side 17 must be diminished in order that the square on it may be 288 *exactly*. If, in accordance with the Indian fashion, a gnomon with unit area were to be subtracted from a square with 17 as side, this would approximately be secured by giving the gnomon the breadth $\frac{1}{34}$, for $2 \times 17 \times \frac{1}{34} = 1$. The side of the smaller square thus arrived at would be $17 - \frac{1}{34} = 12 + 4 + 1 - \frac{1}{34}$, whence, dividing out by 12, we have

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\,.\,4} - \frac{1}{3\,.\,4\,.\,34}, \text{ approximately.}$$

But it is a far cry from this calculation of an approximate value to the discovery of the *irrational*. First, we ask, is there any sign that this value was known to be inexact? It comes directly after the statement (I. 6) that the square on the diagonal of a square is double that of that square, and the rule is quite boldly stated without any qualification: "lessen the unit by one-third and the latter by one-quarter of itself less one-thirty-fourth of this part." Further the approximate value is actually used for the purpose of constructing a square when the side is given (II. 1). So familiar was the formula that it was apparently made the basis of a sub-division of measures of length. Thibaut observes (*Journal of the Asiatic Society of Bengal*, XLIX., p. 241) that, according to Bāudhāyana, the unit of length was divided into 12 *fingerbreadths*, and that one of the two divisions of the *fingerbreadth* was into 34 *sesame-corns*, and he adds that he has no doubt that this division, which he has not elsewhere met, owes its origin to the formula for $\sqrt{2}$. The result of using this subdivision would be that, in a square with side equal to 12 *fingerbreadths*, the diagonal would be 17 *fingerbreadths* less 1 *sesame-corn*. Is it conceivable that a sub-division of a measure of length would be based on an evaluation known to be inexact? No doubt the first discover would be aware that the area of a gnomon with breadth $\frac{1}{34}$ and outer side 17 is not exactly equal to 1 but less than it by the square of $\frac{1}{34}$ or by $\frac{1}{1156}$; as, however, the object of the whole proceeding was purely practical, he would, without hesitation, ignore this as being of no practical importance, and, thereafter, the formula would be handed down and taken as a matter of course without arousing suspicion as to its accuracy. This supposition is confirmed by reference to the sort of rules which the Indians allowed themselves to regard as accurate. Thus Āstamba himself gives a construction for a circle equal in area to a given square, which is equivalent to taking $\pi = 3{\cdot}09$, and yet observes that it gives the required circle "*exactly*" (III. 2), while his construction of a square equal to a circle, which he equally calls "exact," makes the side of the square equal to $\frac{13}{15}$ths of the diameter of the circle (III. 3), and is equivalent to taking $\pi = 3.00\dot{4}$. But, even if some who used the approximation for $\sqrt{2}$ were conscious that it was not quite accurate (of which there is no evidence), there is an immeasurable difference between arrival at this consciousness and the discovery of the irrational. As Vogt says, three stages had to be passed through before the irrationality of the diagonal of a square was discovered in any real sense. (1) All values found by direct measurement or calculations based thereon have to be recognised as being inaccurate. Next (2) must supervene the conviction that it is *impossible* to arrive at an accurate arithmetical expression of the value. And lastly (3) the impossibility must be proved. Now there is no real evidence that the Indians, at the date in question, had even reached the first stage, still less the second or third.

The net results then of Bürk's papers and of the criticisms to which they have given rise appear to be these. (1) It must be admitted that Indian geometry had reached the stage at which we find it in Āpastamba quite independently of Greek influence. But (2) the old Indian geometry was purely empirical and practical, far removed from abstractions such as the irrational. The Indians had indeed by trial in particular cases, persuaded themselves of the truth of the Pythagorean theorem and enunciated it in all its generality; but they had not established it by scientific proof.

**Alternative proofs.**

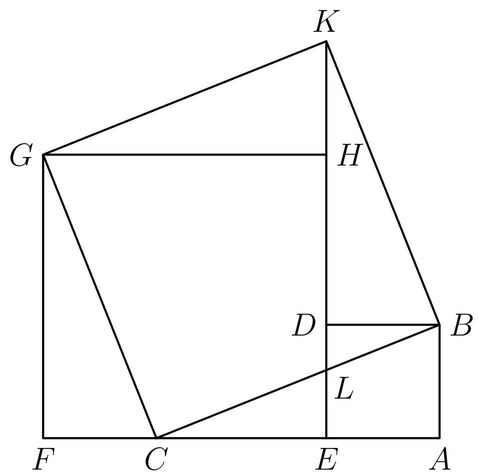
I. The well-known proof of I. 47 obtained by putting two squares side by side, with their bases continuous, and cutting off right-angled triangles which can then be put on again in different positions, is attributed by an-Nairīzī to Thābit b. Qurra (826–901 A.D.).

His actual construction proceeds thus.

Let $ABC$ be the given triangle right-angled at $A$.

Construct on $AB$ the square $AD$;
produce $AC$ to $F$ so that $EF$ may be equal to $AC$.

Construct on $EF$ the square $EG$, and produce $DH$ to $K$ so that $DK$ may be equal to $AC$.




It is then proved that, in the triangles $BAC$, $CFG$, $KHG$, $BDK$,

the sides $BA$, $CF$, $KH$, $BD$ are all equal, and
the sides $AC$, $FG$, $HG$, $DK$ are all equal.

The angles included by the equal sides are all right angles; hence the four triangles are equal in all respects. [I. 4]

Hence $BC$, $CG$, $GK$, $KB$ are all equal.

Further the angles $DBK$, $ABC$ are equal;
hence, if we add to each the angle $DBC$,
the angle $KBC$ is equal to the angle $ABD$ and is therefore a right angle.

In the same way the angle $CGK$ is right;
therefore $BCGK$ is a square, i.e. the square on $BC$.

Now the sum of the quadrilateral $GCLH$ and the triangle $LDB$ together with two of the equal triangles make the squares on $AB$, $AC$, and together with the other two make the square on $BC$.

Therefore etc.

II. Another proof is easily arrived at by taking the particular case of Pappus' more general proposition given below in which the given triangle is right-angled and the parallelograms on the sides containing the right angles are squares. If the figure is drawn, it will be seen that, with no more than one additional line inserted, it contains Thābit's figure, so that Thābit's proof may have been practically derived from that of Pappus.

III. The most interesting of the remaining proofs seems to be that shown in the accompanying figure. It is given by J. W. Müller, *Systematische Zusammenstellung der wichtigsten bisher bekannten Beweise des Pythag. Lehrsatzes* (Nürnberg, 1819), and in the second edition (Mainz, 1821) of Ign. Hoffmann, *Der Pythag. Lehrsatz mit 32 theils bekannten theils neuen Beweisen* [3 more in second edition]. It appears to come from one of the scientific papers of Leonardo da Vinci (1452–1519).

The triangle $HKL$ is constructed on the base $KH$ with the side $KL$ equal to $BC$ and the side $LH$ equal to $AB$.

Then the triangle $HLK$ is equal in all respects to the triangle $ABC$, and to the triangle $EBF$.

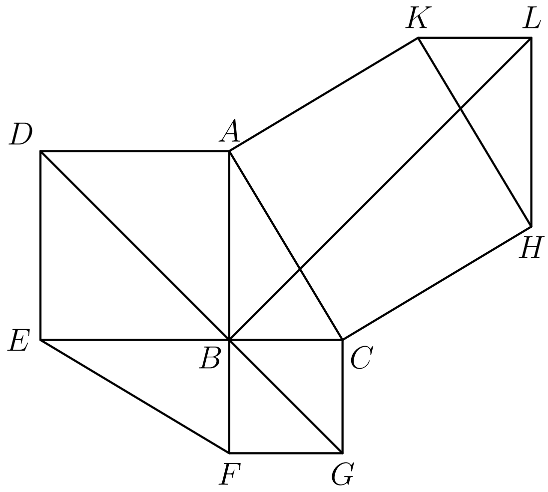
Now $DB$, $BG$, which bisect the angles $ABE$, $CBF$ respectively, are in a straight line. Join $BL$.

It is easily proved that the four quadrilaterals $ADGC$, $EDGF$, $ABLK$, $HLBC$ are all equal.

Hence the hexagons $ADEFGC$, $ABCHLK$ are equal.

Subtracting from the former the two triangles $ABC$, $EBF$, and from the latter the two equal triangles $ABC$, $HLK$, we prove that

the square $CK$ is equal to the sum of the squares $AE$, $CF$.



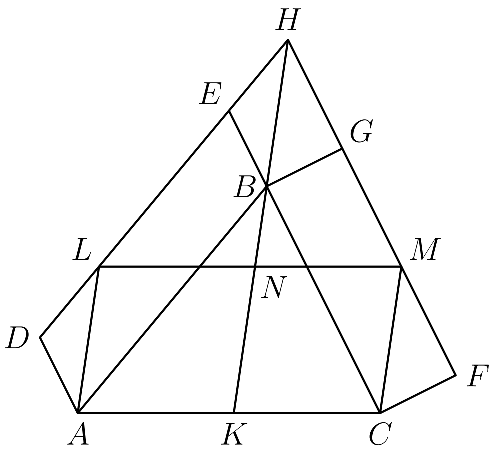

**Pappus' extension of I. 47.**

In this elegant extension the triangle may be *any* triangle (not necessarily right-angled), and *any* parallelograms take the place of squares on two of the sides.

Pappus (IV. 177) enunciates the theorem as follows:

*If* ABC *be a triangle, and any parallelograms whatever* ABED, BCFG *be described on* AB, BC, *and if* DE, FG *be produced to* H, *and* HB *be joined, the parallelograms* ABED, BCFG *are equal to the parallelogram contained by* AC, HB *in an angle which is equal to the sum of the angles* BAC, DHB.

Produce $HB$ to $K$; through $A$, $C$ draw $AL$, $CM$ parallel to $HK$, and join $LM$.




Then, since $ALHB$ is a parallelogram, $AL$, $HB$ are equal and parallel. Similarly $MC$, $HB$ are equal and parallel.

Therefore $AL$, $MC$ are equal and parallel;
whence $LM$, $AC$ are also equal and parallel,

and $ALMC$ is a parallelogram.

Further, the angle $LAC$ of this parallelogram is equal to the sum of the angles $BAC$, $DHB$, since the angle $DHB$ is equal to the angle $LAB$.

Now, since the parallelogram $DABE$ is equal to the parallelogram $LABH$ (for they are on the same base $AB$ and in the same parallels $AB$, $DH$), and likewise $LABH$ is equal to $LAKN$ (for they are on the same base $LA$ and in the same parallels $LA$, $HK$),

the parallelogram $DABE$ is equal to the parallelogram $LAKN$.

For the same reason,

the parallelogram $BGFC$ is equal to the parallelogram $NKCM$.

Therefore the sum of the parallelograms $DABE$, $BGFC$ is equal to the parallelogram $LACM$, that is, to the parallelogram which is contained by $AC$, $HB$ in an angle $LAC$ which is equal to the sum of the angles $BAC$, $BHD$.

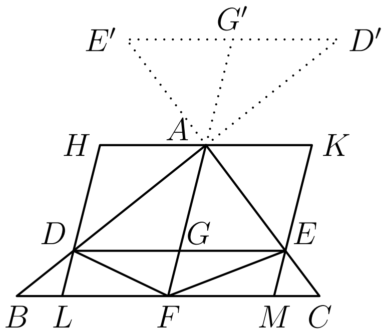
"And this is far more general than what is proved in the Elements about squares in the case of right-angled (triangles)."

**Heron's proof that AL, BK, CF in Euclid's figure meet in a point.**

The final words of Proclus's note on I. 47 (p. 429, 9–15) are historically interesting. He says: "The demonstration by the writer of the Elements being clear, I consider that it is unnecessary to add anything further, and that we may be satisfied with what has been written, since in fact those who have added anything more, like Pappus and Heron, were obliged to draw upon what is proved in the sixth Book, for no really useful object." Those words cannot of course refer to the extension of I. 47 given by Pappus; but the key to them, so far as Heron is concerned, is to be found in the commentary of an-Nairīzī (pp. 175–185, ed. Besthorn-Heiberg; pp. 78–84, ed. Curtze) on I. 47, wherein he gives Heron's proof that the lines $AL$, $FC$, $BK$ in Euclid's figure meet in a point. Heron proved this by means of three lemmas which would most naturally be proved from the principle of similitude as laid down in Book VI., but which Heron, as a *tour de force*, proved on the principles of Book I only. The *first* lemma is to the following effect.

*If, in a triangle* ABC, DE *be drawn parallel to the base* BC*, and if* AF *be drawn from the vertex* A *to the middle point* F *of* BC*, then* AF *will also bisect* DE.

This is proved by drawing $HK$ through $A$ parallel to $DE$ or $BC$, and $HDL$, $KEM$ through $D$, $E$ respectively parallel to $AGF$, and lastly joining $DF$, $EF$.




Then the triangles $ABF$, $AFC$ are equal (being on equal bases), and the triangles $DBF$, $EFC$ are also equal (being on equal bases and between the same parallels).

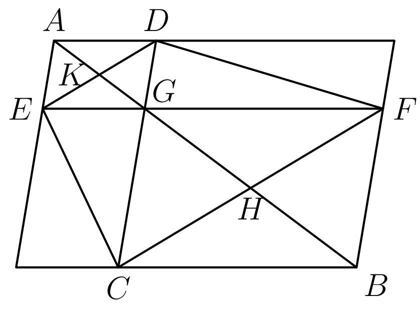
Therefore, by subtraction, the triangles $ADF$, $AEF$ are equal, and hence the parallelograms $AL$, $AM$ are equal.

These parallelograms are between the same parallels $LM$, $HK$; therefore $LF$, $FM$ are equal, whence $DG$, $GE$ are also equal.

The *second* lemma is an extension of this to the case where $DE$ meets $BA$, $CA$ produced beyond $A$.

The *third* lemma proves the converse of Euclid I. 43, that, *If a parallelogram AB is cut into four others* ADGE*,* DF*,* FGCB*,* CE*, so that* DF*,* CE *are equal, the common vertex* G *will be on the diagonal* AB*.*

Heron produces $AG$ till it meets $CF$ in $H$. Then, if we join $HB$, we have to prove that $AHB$ is one straight line. The proof is as follows. Since the areas $DF$, $EC$ are equal, the triangles $DFG$, $ECG$ are equal.




If we add to each the triangle $CGF$,

the triangles $ECF$, $DCF$ are equal;

therefore $ED$, $CF$ are parallel.

Now it follows from I. 34, 29 and 26 that the triangles $AKE$, $GKD$ are equal in all respects;

therefore $EK$ is equal to $KD$.

Hence, by the second lemma,

$CH$ is equal to $HF$.

Therefore, in the triangles $FHB$, $CHG$,
the two sides $BF$, $FH$ are equal to the two sides $GC$, $CH$,

and the angle $BFH$ is equal to the angle $GCH$;
hence the triangles are equal in all respects.
and the angle $BHF$ is equal to the angle $GHC$.

Adding to each the angle $GHF$, we find that the angles $BHF$, $FHG$ are equal to the angles $CHG$, $GHF$,

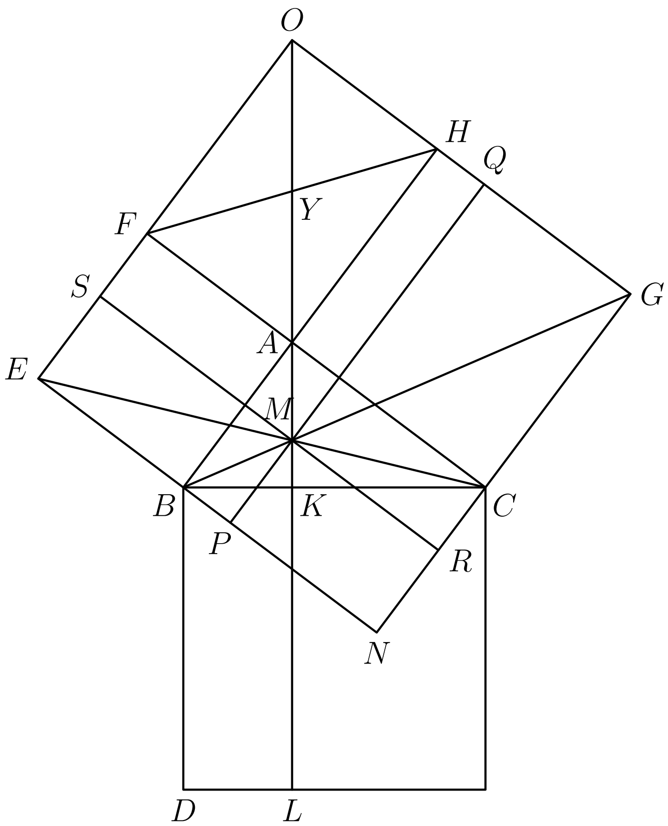
and therefore to two right angles.

Therefore $AHB$ is a straight line.

Heron now proceeds to prove the proposition that, in the accompanying figure, if $AKL$ perpendicular to $BC$ meet $EC$ in $M$, and if $BM$, $MG$ be joined,

$BM$, $MG$ are in one straight line.

Parallelograms are completed as shown in the figure, and the diagonals $OA$, $FH$ of the parallelogram $FH$ are drawn.




Then the triangles $FAH$, $BAC$ are clearly equal in all respects;

therefore the angle $HFA$ is equal to the angle $ABC$, and therefore to the angle $CAK$ (since $AK$ is perpendicular to $BC$).

But, the diagonals of the rectangle $FH$ cutting one another in $Y$,

$FY$ is equal to $YA$,

and the angle $HFA$ is equal to the angle $OAF$.

Therefore the angles $OAF$, $CAK$ are equal, and accordingly

$OA$, $AK$ are in a straight line.

Hence $OM$ is the diagonal of $SQ$;

therefore $AS$ is equal to $AQ$,

and, if we add $AM$ to each,

$FM$ is equal to $MH$.

But, since $EC$ is the diagonal of the parallelogram $FN$,

$FM$ is equal to $MN$.

Therefore $MH$ is equal to $MN$;

and, by the third lemma, $BM$, $MG$ are in a straight line.