[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 342–345 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 44.]

14. **since the straight line HF falls**. . . . The verb is in the aorist (ἐνέπεσεν) here and in similar expressions in the following propositions.

This proposition will always remain one of the most impressive in all geometry when account is taken (1) of the great importance of the result obtained, the transformation of a parallelogram of any shape into another with the same angle and of equal area but with one side of any given length, e.g., a *unit* length, and (2) of the simplicity of the means employed, namely the mere application of the property that the complements of the "parallelograms about the diameter" of a parallelogram are equal. The marvellous ingenuity of the solution is indeed worthy of the "godlike men of old," as Proclus calls the discovers of the method of "application of areas"; and there would seem to be no reason to doubt that the particular solution, like the whole theory, was Pythagorean, and not a new solution due to Euclid himself.

Application of areas.

On this proposition Proclus gives (pp. 419, 15–420, 23) a valuable note on the method of "appplication of areas" here introduced, which was one of the most powerful methods on which Greek geometry relied. The note runs as follows:

"These things, says Eudemus (οἱ περὶ τὸν Εὔδημον), are ancient and are discoveries of the Muse of the Pythagoreans, I mean the *application of areas* (παραβολὴ τῶν χωρίων), their *exceeding* (ὑπερβολή) and their *falling-short* (ἔλλειψις). It was from the Pythagoreans that later geometers [i.e. Apollonius] took the names, which they again transferred to the so-called *conic* lines, designating one of these a *parabola* (application), another a *hyperbola* (exceeding) and another an *ellipse* (falling-short), whereas those godlike men of old saw the things signified by these names in the construction, in a plane, of areas upon a finite straight line. For, when you have a straight line set out and lay the given area exactly alongside the whole of the straight line, then they say that you *apply* (παραβάλλειν) the said area; when however you make the length of the area greater than the straight line itself, it is said to *exceed* (ὑπερβάλλειν), and when you make it less, in which case, after the area has been drawn, there is some part of the straight line extending beyond it, it is said to *fall short* (ἐλλείπειν). Euclid too, in the sixth book, speaks in this way both of *exceeding* and *falling-short*; but in this place he needed the *application* simply, as he sought to apply to a given straight line an area equal to a given triangle in order that we might have in our power, not only the *construction* (σύστασις) of a parallelogram equal to a given triangle, but also the *application* of it to a finite straight line. For example, given a triangle with an area of 12 feet, and a straight line set out the length of which is 4 feet, we apply to the straight line the area equal to the triangle if we take the whole length of 4 feet and find out how many feet the breadth must be in order that the parallelogram may be equal to the triangle. In the particular case, if we find a breadth of 3 feet and multiply the length into the breadth, supposing that the angle set out is a right angle, we shall have the area. Such then is the *application* handed down from early times by the Pythagoreans."

Other passages to a similar effect are quoted from Plutarch. (1) "Pythagoras sacrificed an ox on the strength of his proposition (διάγραμμα) as Apollodotus (?-rus) says ... whether it was the theorem of the hypotenuse, viz. that the square on it is equal to the squares on the sides containing the right angle, or the problem about the *application of an area*." (*Non posse suaviter vivi secundum Epicurum*, c. 11.) (2) "Among the most geometrical theorems, or rather problems, is the following: given two figures, to *apply* a third equal to the one and similar to the other, on the strength of which discovery they say moreover that Pythagoras sacrificed. This is indeed unquestionably more subtle and more scientific than the theorem which demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle" (*Symp.* VIII. 2, 4).

The story of the sacrifice must (as noted by Bretschneider and Hankel) be given up as inconsistent with Pythagorean ritual, which forbade such sacrifices; but there is no reason to doubt that the first distinct formulation and introduction into Greek geometry of the method of *application of areas* was due to the Pythagoreans. The complete exposition of the *application* of areas, their *exceeding* and their *falling-short*, and of the construction of a rectilineal figure equal to one given figure and similar to another, takes us into the sixth Book of Euclid; but it will be convenient to note here the general features of the theory of *application*, *exceeding* and *falling-short.*

The simple *application* of a parallelogram of given area to a given straight line as one of its sides is what we have in I. 44 and 45; the general form of the problem with regard to *exceeding* and *falling-short* may be stated thus:

"To apply to a given straight line a rectangle (or, more generally, a parallelogram) equal to a given rectilineal figure and (1) *exceeding* or (2) *falling-short* by a square (or, in the more general case, a parallelogram similar to a given parallelogram)."

What is meant by saying that the applied parallelogram (1) *exceeds* or (2) *falls short* is that, while its base coincides and is coterminous *at one end* with the straight line, the said base (1) overlaps or (2) falls short of the straight line *at the other end*, and the portion by which the applied parallelogram exceeds a parallelogram of the same angle and height on the given straigh line (*exactly*) as base is a parallelogram similar to a given parallelogram (or, in particular cases, a square). In the case where the parallelogram is to *fall short*, a διορισμός is necessary to express the condition of possibility of solution.

We shall have occasion to see, when we come to the relative propositions in the second and sixth Books, that the general problem here stated is equivalent to that of solving geometrically a mixed quadratic equation. We shall see that, even by means of II. 5 and 6, we can solve geometrically the equations

$$\begin{aligned} ax \pm x^2 &= b^2, \\ x^2 - ax &= b^2; \end{aligned}$$

but in VI. 28, 29 Euclid gives the equivalent of the solution of the general equations

$$ax \pm \frac{b}{c}x^2 = \frac{C}{m}.$$

We are now in a position to understand the application of the terms *parabola* (application), *hyperbola* (exceeding) and *ellipse* (falling-short) to conic sections. These names were first so applied by Apollonius as expressing in each case the fundamental property of the curves as stated by him. This fundamental property is the geometrical equivalent of the Cartesian equation referred to any diameter of the conic and the tangent at its extremity as (in general oblique) axes. If the *parameter* of the ordinates from the several points of the conic drawn to the given diameter be denoted by p (p being accordingly, in the case of the hyperbola and ellipse, equal to $\frac{d'^2}{d^2}$, where d is the length of the given diameter and d' that of its conjugate), Apollonius gives the properties of the three conics in the following form.

(1) For the *parabola*, the square on the ordinate at any point is equal to a rectangle applied to p as base with altitude equal to the corresponding abscissa. That is to say, with the usual notation,

$$y^2 = px.$$

(2) For the *hyperbola* and *ellipse*, the square on the ordinate is equal to the rectangle applied to p having as its width the abscissa and *exceeding* (for the hyperbola) or *falling-short* (for the ellipse) by a figure similar and similarly situated to the rectangle contained by the given diameter and p.

That is, in the *hyperbola* $y^2 = px + \dfrac{x^2}{d^2}\, pd,$

or $y^2 = px + \dfrac{p}{d}\, x^2,$

and in the *ellipse* $y^2 = px - \dfrac{p}{d}\, x^2.$

The form of these equations will be seen to be exactly the same as that of the general equations above given, and thus Apollonius' nomenclature followed exactly the traditional theory of *application*, *exceeding*, and *falling-short*.