[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 334–336 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 38.]

On this proposition Proclus remarks (pp. 405–6) that Euclid seems to him to have given in VI. 1 one proof including all the four theorems from I. 35 to I. 38, and that most people have failed to notice this. When Euclid, he says, proves that triangles and parallelograms of the same altitude have to one another the same ratio as their bases, he simply proves all these propositions more generally by the use of proportion: for of course to be of the same altitude is equivalent to being in the same parallels. It is true that VI. 1 generalises these propositions, but it must be observed that it does not prove these propositions themselves, as Proclus seems to imply; they are in fact assumed in order to prove VI. 1.

Comparison of areas of triangles of I. 24.

The theorem already mentioned as given by Proclus on I. 24 (pp. 340–4) is placed here by Heron, who also enunciates it more clearly (an-Nairīzī, ed. Besthorn-Heiberg, pp. 155–161, ed. Curtze, pp. 75–8).

If in two triangles two sides of the one be equal to two sides of the other respectively, and the angle of the one be greater than the angle of the other, namely the angles contained by the equal sides, then (1) if the sum of the two angles contained by the equal sides is equal to two right angles, the two triangles are equal to one another; (2) if less than two right angles, the triangle which has the greater angle is also itself greater than the other; (3) if greater than two right angles, the triangle which has the less angle is greater than the other triangle.

Let two triangles ABC, DEF have the sides AB, AC respectively equal to DE, DF.

(1) First, suppose that the angles at A and D in the triangles ABC, DEF are together equal to two right angles.

Heron's construction is now as follows.

Make the angle EDG equal to the angle BAC.

Draw FH parallel to ED meeting DG in H.

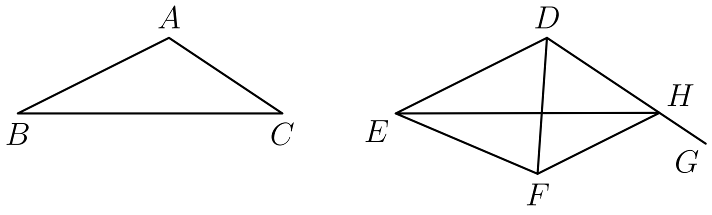
Join EH.

Then, since the angles BAC, EDF are equal to two right angles, the angles EDH, EDF are equal to two right angles.

But so are the angles EDH, DHF.

Therefore the angles EDF, DHF are equal.

And the alternate angles EDF, DFH are equal. [I. 29]



Therefore the angles DHF, DFH are equal.

and DF is equal to DH. [I. 6]

Hence the two sides ED, DH are equal to the two sides BA, AC; and the included angles are equal.

Therefore the triangles ABC, DEH are equal in all respects.

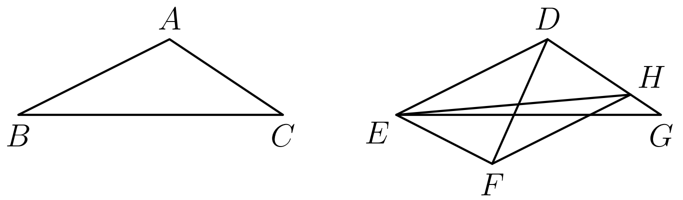
And the triangles DEF, DEH between the same parallels are equal. [I. 37]

Therefore the angles ABC, DEF are equal.

[Proclus takes the construction of Eucl. I. 24, i.e., he makes DH equal to DF and then proves ED, FH are parallel.]

(2) Suppose the angles BAC, EDF together *less* than two right angles.

As before, make the angle EDG equal to the angle BAC, draw FH parallel to ED, and join EH.



In this case the angles EDH, EDF are together less than two right angles, while the angles EDH, DHF are equal to two right angles. [I. 29]

Hence the angle EDF, and therefore the angle DFH, is less than the angle DHF.

Therefore DH is less than DF. [I. 19]

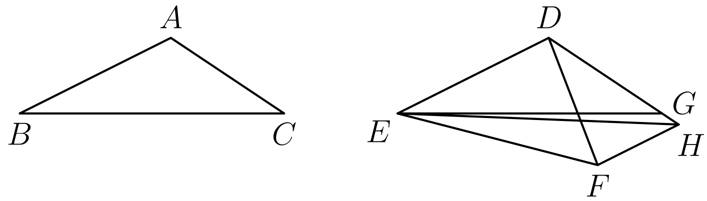
Produce DH to G so that DG is equal to DF or AC, and join EG.

Then the triangle DEG, which is equal to the triangle ABC, is greater than the triangle DEH, and therefore greater than the triangle DEF.

(3) Suppose the angles BAC, EDF together greater than two right angles.

We make the same construction in this case, and we prove in like manner that the angle DHF is less than the angle DFH,

whence DH is greater than DF or AC.

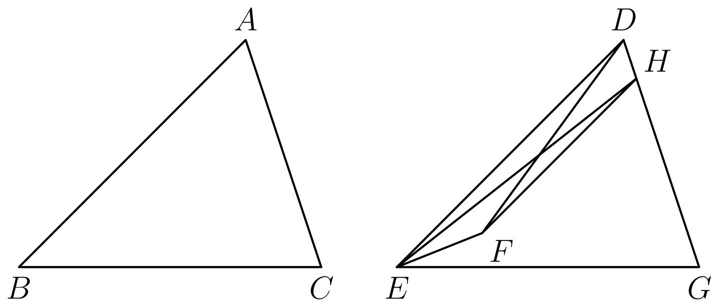


Make DG equal to AC, and join EG.

It then follows that the triangle DEF is greater than the triangle ABC.

[In the second and third cases again Proclus starts from the construction in I. 24, and proves, in the second case, that the parallel, FH, to ED cuts DG and, in the third case, that it cuts DG produced.]

There is no necessity for Heron to take account of the position of F in relation to the side opposite D. For in the first and third cases F *must* fall in the position in which Euclid draws it in I. 24, whatever be the relative lengths of AB, AC.



In the second case the figure *may* be as annexed, but the proof is the same, or rather the case needs no proof at all.