[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 327–331 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 35.]

21. **FDC.** The text has "DFC."

22. **Let DGE be subtracted.** Euclid speaks of the triangle DGE without any explanation that, in the case which he takes (where AD, EF have no point in common), BE, CD must meet at a point G between the two parallels. He allows this to appear from the figure simply.

Equality in a new sense.

It is important to observe that we are in this proposition introduced for the first time to a new conception of equality between figures. Hitherto we have had equality in the sense of *congruence* only, as applied to straight lines, angles, and even triangles (cf. I. 4). Now, without any explicit reference to any change in the meaning of the term, figures are inferred to be *equal* which are equal in *area* or in *content* but need not be of the same *form*. No *definition* of equality is anywhere given in Euclid; we are left to infer its meaning from the few *axioms* about "equal things." It will be observed that in the above proof the "equality" of two parallelograms on the same base and between the same parallels is inferred by the successive steps (1) of subtracting one and the same area (the triangle DGE) from two areas equal in the sense of *congruence* (the triangles AEB, DFC), and inferring that the remainders (the trapezia $ABGD$, $EGCF$) are "equal"; (2) of adding one and the same area (the triangle GBC) to each of the latter "equal" trapezia, and inferring the equality of the respective sums (the two given parallelograms).

As is well known, Simson (after Clairaut) slightly altered the proof in order to make it applicable to all the three possible cases. The alteration substituted *one* step of subtracting congruent areas (the triangles AEB, DFC) from one and the same area (the trapezium $ABCF$) for the *two* steps above shown of first subtracting and then adding a certain area.

While, in either case, nothing more is explicitly used than the axioms that *if equals be added to equals, the wholes are equal* and that, *if equals be subtracted from equals, the remainders are equal*, there is the further *tacit* assumption that it is indifferent to *what part* or from *what part* of the same or equal areas the same or equal areas are added or subtracted. De Morgan observes that the postulate "an area taken from an area leaves the same area form whatever part it may be taken" is particularly important as the key to equality of non-rectilineal areas which could not be cut into coincidence geometrically.

Legendre introduced the word *equivalent* to express this wider sense of equality, restricting the term *equal* to things equal in the sense of congruent; and this distinction has been found convenient.

I do not think it necessary, nor have I the space, to give any account of the recent development of the theory of equivalence on new lines represented by the researches of W. Bolyai, Duhamel, De Zolt, Stolz, Schur, Veronese, Hilbert and others, and must refer the reader to Ugo Amaldi's article *Sulla teoria dell' equivalenza* in *Questioni riguardanti le mathematiche elementari*, I. (Bologna, 1912), pp. 145–198, and to Max Simon, *Über die Entwicklung der Elementar-geometrie im XIX. Jahrhundert* (Leipzig, 1906), pp. 115–120, with their full references to the literature of the subject. I may however refer to the suggestive distinction of phraseology used by Hilbert (*Grundlagen der Geometrie*, pp. 39, 40):

(1) "Two polygons are called *divisibly-equal* (*zerlegungsgleich*) if they can be divided into a *finite* number of triangles which are congruent two and two."

(2) "Two polygons are called *equal in content* (*inhaltsgleich*) or *of equal content* if it is possible to add *divisibly-equal* polygons to them in such a way that the two combined polygons are *divisibly-equal*."

(Amaldi suggests as alternatives for the terms in (1) and (2) the expressions *equivalent by sum* and *equivalent by difference* respectively.)

From these definitions it follows that "by combining *divisibly-equal* polygons we again arrive at *divisibly-equal* polygons; and, if we subtract *divisibly-equal* polygons from *divisibly-equal* polygons, the polygons remaining are *equal in content*."

The proposition also follows without difficulty that "if two polygons are *divisibly-equal* to a third polygon, they are also *divisibly-equal* to one another; and, if two polygons are *equal in content* to a third polygon, they are *equal in content* to one another."

The different cases.

As usual, Proclus (pp. 399–400), observing that Euclid has given only the most difficult of the three possible cases, adds the other two with separate proofs. In the case where E in the figure of the proposition falls between A and D, he *adds* the congruent triangles ABE, DCF respectively to the smaller trapezium $EBCD$, instead of subtracting them (as Simson does) from the larger trapezium $ABCF$.

An ancient "Budget of Paradoxes."

Proclus observes (p. 396, 12 sqq.) that the present theorem and the similar one relating to triangles are among the so-called paradoxical theorems of mathematics, since the uninstructed might well regard it as impossible that the area of the parallelograms should remain the same while the length of the sides other than the base and the side opposite to it may increase indefinitely. He adds that mathematicians had made a collection of such paradoxes, the so-called *treasury of paradoxes* (ὁ παράδοξος τόπος)—cf. the similar expressions τόπος ἀναλυόμενος (treasury of analysis) and τόπος ἀστρονομούμενος—in the same way as the Stoics with their *illustrations* (ὥσπερ οἱ ἀπὸ τῆς Στοᾶς ἐπὶ τῶν δειγμάτων). It may be that this *treasury of paradoxes* was the work of Erycinus quoted by Pappus (III. p. 107, 8) and mentioned above (note on I. 21, p. 290).

Locus-theorems and loci in Greek geometry.

The proposition I. 35 is, says Proclus (394–6) the first *locus-theorem* (τοπικὸν θεώρημα) given by Euclid. Accordingly it is in his note on this proposition that Proclus gives us his view of the nature of a locus-theorem and of the meaning of the word *locus* (τόπος); and great importance attaches to his words because he is one of the three writers (Pappus and Eutocius being the two others) upon whom we have to rely for all that is known of the Greek conception of geometrical loci.

Proclus' explanation (pp. 394, 15–395, 2) is as follows. "I call those (theorems) *locus-theorems* (τοπικά) in which the same property is found to exist on the whole of some locus (πρὸς ὅλῳ τινὶ τόπῳ), and (I call) a locus a position of a line or a surface producing one and the same property (γραμμῆς ἢ ἐπιφανείας θέσιν ποιοῦσαν ἓν καὶ ταὐτὸν σύμπτωμα). For, of locus-theorems, some are constructed on lines and others on surfaces (τῶν γὰρ τοπικῶν τὰ μέν ἐστι πρὸς γραμμαῖς συνιστάμενα, τὰ δὲ πρὸς ἐπιφανείαις). And, since some lines are plane (ἐπίπεδοι) and others solid (στερεαί)—those being plane which are simply conceived of in a plane (ὧν ἐν ἐπιπέδῳ ἁπλῆ ἡ νόησις), and those solid the origin of which is revealed from some section of a solid figure, as the cylindrical helix and the conic lines (ὡς τῆς κυλινδρικῆς ἕλικος καὶ τῶν κωνικῶν γραμμῶν)—I should say (φαίην ἄν) further that, of locus-theorems on lines, some give a plane locus and others a solid locus."

Leaving out of sight for the moment the class of *loci on surfaces*, we find that the distinction between *plane* and *solid loci*, or *plane* and *solid lines*, was similarly understood by Eutocius, who says (Apollonius, ed. Heiberg, II. p. 184) that "*solid loci* have obtained their name from the fact that the lines used in the solution of problems regarding them have their origin in the section of solids, for example the sections of the cone and several others." Similarly we gather from Pappus that *plane loci* were straight lines and circles, and *solid loci* were conics. Thus he tells us (VII. p. 672, 20) that Aristaeus wrote five books of *Solid Loci* "supplementary to (literally, continuous with) the conics"; and, though Hultsch brackets the passage (VII. p. 662, 10–15) which says plainly that *plane loci* are straight lines and circles, while *solid loci* are sections of cones, i.e. parabolas, ellipses and hyperbolas, we have the exactly corresponding distinction drawn by Pappus (III. p. 54, 7–16) between *plane* and *solid problems*, plane problems being those solved by means of straight lines and circumferences of circles, and solid problems those solved by means of one or more of the sections of the cone. But, whereas Proclus and Eutocius speak of other *solid loci* besides conics, there is nothing in Pappus to support the wider application of the term. According to Pappus (III. p. 54, 16–21) problems which could not be solved by means of striaght lines, circles, or conics were *linear* (γραμμικά) because they used for their construction lines having a more complicated and unnatural origin than those mentioned, namely such curves as *quadratrices*, conchoids and cissoids. Similarly, in the passage supposed to be interpolated, *linear loci* are distinguished as those which are neither straight lines nor circles nor any of the conic sections (VII. p. 662, 13–15). Thus the classification given by Proclus and Eutocius is less precise than that which we find in Pappus: and the inclusion by Proclus of the cylindrical helix among solid loci, on the ground that it arises from a section of a solid figure, would seem to be, in any case, due to some misapprehension.

Comparing these passages and the hints in Pappus about *loci on surfaces* (τόποι πρὸς ἐπιφανείᾳ) with special reference to Euclid's two books under that title, Heiberg concludes that *loci on lines* and *loci on surfaces* in Proclus' explanation are loci which *are* lines and loci which *are* surfaces respectively. But some qualification is necessary as regards Proclus' conception of *loci on lines*, because he goes on to say (p. 395, 5), with reference to this proposition, that, while the locus is a *locus on lines* and moreover *plane*, it is "the whole space between the parallels" which is the locus of the various parallelograms on the same base proved to be equal in area. Similarly, when he quotes III. 21 about the equality of the angles in the same segment and III. 31 about the right angle in a semicircle as cases where a circumference of a circle takes the place of a straight line in a *plane* locus-theorem, he appears to imply that it is the segment or semicircle as an *area* which is regarded as the locus of an infinite number of *triangles* with the same base and equal vertical angles, rather than that it is the *circumference* which is the locus of the angular *points*. Likewise he gives the equality of the parallelograms inscribed in "the asymptotes and the hyperbola" as an example of a *solid* locus-theorem, as if the area included between the curve and its asymptotes was regarded as the *locus* of the equal parallelograms. However this may be, it is clear that the locus in the present proposition can only be either (1) a *line*-locus of a *line*, not a point, or (2) and *area*-locus of an *area*, not a point or a line;

and we seem to be thus brought to another and different classification of loci corresponding to that quoted by Pappus (VII. p. 660, 18 sqq.) from the preliminary exposition given by Apollonius in his *Plane Loci*. According to this, loci in general are of three kinds: (1) ἐφεκτικοί, *holding-in*, in which sense the locus of a point is a point, of a line a line, of a surface a surface, and of a solid a solid, (2) διεξοδικοί, *moving along*, a line being in this sense a locus of a point, a surface of a line and a solid of a surface, (3) ἀναστροφικοί, where a surface is a locus of a point and a solid of a line. Thus the locus in this proposition, whether it is the space between the two parallels regarded as the locus of the equal parallelograms, or the line parallel to the base regarded as the locus of the sides opposite to the base, would seem to be of the first class (ἐφεκτικός); and, as Proclus takes the former view of it, a *locus on lines* also, the locus being *plane* in the particular case of III. 21, 31, by straight lines and circles, but not by any higher curves.

Proclus notes lastly (p. 395, 13–21) that, according to Geminus, "Chrysippus likened locus-theorems to the *ideas*. For, as the ideas confine the genesis of unlimited (particulars) within defined limits, so in such theorems the unlimited (particular figures) are confined within defined *places* or *loci* (τόποι). And it is this boundary which is the cause of the equality; for the height of the parallels, which remains the same, while an infinite number of parallelograms are conceived on the same base, is what makes them all equal to one another."