[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 317–322 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 32.]

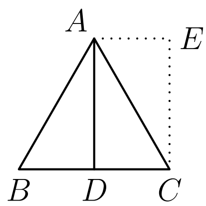
This theorem was discovered in the very early stages of Greek geometry. What we know of the history of it is gathered from three allusions found in Eutocius, Proclus and Diogenes Laertius respectively.

1. Eutocius at the beginning of his commentarty on the *Conics* of Apollonius (ed. Heiberg, Vol. II. p. 170) quotes Geminus as saying that "the ancients (οἱ ἀρχαῖοι) investigated the theorem of the two right angles in each individual species of triangle first in the equilateral, again in the isosceles, and afterwards in the the scalene triangle, and later geometers demonstrated the general theorem ot the effect that in *any* triangle the three interior angles are equal to two right angles."

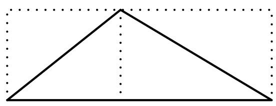
2. Now, according to Proclus (p. 379, 2–5), Eudemus the Peripatetic refers the discovery of this theorem to the Pythagoreans and gives what he affirms to be their demonstration of it. This demonstration will be given below, but it should be remarked that it is general, and therefore that the "later geometers" spoken of by Geminus were presumably the Pythagoreans, whence it appears that the "ancients" contrasted with them must have belonged to the time of Thales, if they were not his Egyptian instructors.

3. That the truth of the theorem was known to Thales might also be inferred from the statement of Pamphile (quoted by Diogenes Laertius, I. 24–5, p. 6, ed. Cobet) that "he, having learnt geometry from the Egyptians, was the first to inscribe a right-angled triangle in a circle and sacrificed an ox" (on the strength of it); in other words, he discovered that the angle in a semicircle is a right angle. No doubt, when this fact was once discovered (*empirically*, say), the consideration of the two isosceles triangles having the centre for vertex and the sides of the right angle for bases respectively, with the help of the theorem of Eucl. I. 5, also known to Thales, would easily lead to the conclusion that the sum of the angles of a *right-angled* triangles is equal to two right angles, and it could be readily inferred that the angles of *any* triangle were likewise equal to two right angles (by resolving it into two right-angled triangles). But it is not easy to see how the property of the angle in a semicircle could be *proved* except (in the reverse order) by means of the equality of the sum of the angles of a *right-angled* triangle to two right angles; and hence it is most natural to suppose, with Cantor, that Thales proved it (if he did prove it) practically as Euclid does in III. 31, i.e., by means of I. 32 as applied to *right-angled* triangles at all events.

If the theorem of I. 32 was proved before Thales' time, or by Thales himself, by the stages indicated in the note of Geminus, we may be satisfied that the reconstruction of the argument of the older proof by Hankel (pp. 96–7) and Cantor (I_3, pp. 143–4) is not far wrong. First, it must have been observed that six angles equal to an angle of an equilateral triangle would, if placed adjacent to one another round a common vertex, fill up the whole space round that vertex. It is true that Proclus attributes to the Pythagoreans the general theorem that only three kinds of regular polygons, the equilateral triangle, the square and the regular hexagon, can fill up the entire space round a point, but the practical knowledge that equilateral triangles have this property could hardly have escaped the Egyptians, whether they made floors with tiles in the form of equilateral triangles or regular hexagons (Allman, *Greek Geometry from Thales to Euclid*, p. 12) or joined the ends of adjacent radii of a figure like the six-spoked wheel, which was their common form of wheel from the time of Ramses II. of the nineteenth Dynasty, say 1300 B.C. (Cantor, I_3, p. 109). It would then be clear that six angles equal to an angle of an equilateral triangle are equal to two right angles. (It would be as clear or clearer, from observation of a square divided into two triangles by a diagonal, that an isosceles right-angled triangle has each of its equal angles equal to half a right angle, so that an isosceles right-angled triangle must have the sum of its angles equal to two right angles.) Next, with regard to the equilateral triangle, it could not fail to be observed that if AD were drawn from the vertex A perpendicular to the base BC, each of the two right-angled triangles so formed would have the sum of its angles equal to two right angles; and this would be confirmed by completing the rectangle $ADCE$, when it would be seen that the rectangle (with its angles equal to four right angles) was divided by its diagonal into two equal triangles, each of which had the sum of its angle equal to two right angles. Next it would be



inferred, as the result of drawing the diagonal of *any* rectangle and observing the equality of the triangles forming the two halves, that the sum of the angles of *any* right-angled triangle is equal to two right angles, and hence (the two congruent right-angled triangles being then placed so as to form one isosceles triangle) that the same is true of *any isosceles* triangle. Only the last step remained, namely that of observing that any triangle could be regarded as the half of a rectangle (drawn as indicated in the next figure), or simply that any triangle could be divided into two right-angled triangles, whence it would be inferred that in general the sum of the angles of any triangle is equal to two right angles.



Such would be the probabilities if we could absolutely rely on upon the statements attributed to Pamphile and Geminus respectively. But in fact there is considerable ground for doubt in both cases.

1. Pamphile's story of the sacrifice of an ox by Thales for joy at his discovery that the angle in a semicircle is a right angle is too suspiciously like the similar story told with reference to Pythagoras and his discovery of the theorem of Eucl. I. 47 (Proclus, p. 426, 6–9). And, as if this were not enough, Diogenes Laertius immediately adds that "others, among whom is Apollodorus the calculator (ὁ λογιστικός), say it was Pythagoras" (sc. who "inscribed the right-angled triangle in a circle"). Now Pamphile lived in the reign of Nero (A.D. 54–68) and therefore some 700 years after the birth of Thales (about 640 B.C.). I do not know on what Max Schmidt bases his statement (*Kulturhistorische Beiträge zur Kenntnis des griechischen und römischen Altertums*, 1906, p. 31) that "other, *much older*, sources name Pythagoras as the discoverer of the said proposition," because nothing more seems to be known of Apollodorus than what is stated here by Diogenes Laertius. But it would at least appear that Apollodorus was only one of several authorities who attributed the proposition to Pythagoras, while Pamphile is alone mentioned as referring it to Thales. Again, the connexion of Pythagoras with the investigation of the right-angled triangle makes it *a priori* more likely that it would be he who would discover its relation to a semicircle. On the whole, therefore the attribution to Thales would seem to be more than doubtful.

2. As regards Geminus' account of the three stages through which the proof of the theorem of I. 32 passed, we note, first, that it is certainly not confirmed by Eudemus, who referred to the Pythagoreans the *discovery* of the theorem that hte sum of the angles of *any* triangle is equal to two right angles and says nothing about any gradual stages by which it was proved. Secondly, it must be admitted, I think, that in the evolution of the proof as reconstructed by Hankel the middle stage is rather artificial and unnecessary, since, once it is proved that *any right-angled* triangle has the sum of its angles equal to two right angles, it is just as easy to pass at once to any *scalene* triangle (which is decomposable into two *unequal* right-angled triangles) as to the isosceles triangle made up of two congruent right-angled triangles.

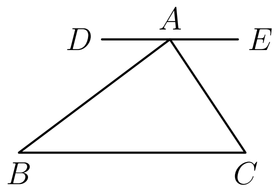
Thirdly, as Heiberg has recently pointed out (*Mathematisches zu Aristoteles*, p. 20), it is quite possible that the statement of Geminus from beginning to end is simply due to a misapprehension of a passage of Aristotle (*Anal. Post.* I. 5, 74 a 25). Aristotle is illustrating his contention that a property is not scientifically proved to belong to a class of things unless it is proved to belong *primarily* (πρῶτον) and *generally* (καθόλου) to the *whole* of the class. His first illustration relates to parallels making with a transversal angles on the same side together equal to two right angles, and has been quoted above in the note on I. 27 (pp. 308–9). His second illustration refers to the transformation of a proportion *alternando*, which (he says) "used at one time to be proved separately" for numbers, lines, solids and times, although it admits of being proved of all at once by one demonstration. The third illustration is: "For the same reason, even *if one should prove* (οὐδ᾽ ἄν τις δείξῃ) with reference to each (sort of) triangle, the equilateral, scalene and isosceles, separately, that each has its angles equal to two right angles, either by one proof or by different proofs, he does not yet know that *the triangle*, i.e. the triangle *in general*, has its angles equal to two right angles, except in a sophistical sense, even though there exists no triangle other than triangles of the kinds mentioned. For he knows it, not *quã* triangle, nor of *every* triangle, except in a numerical sense (κατ᾽ ἀριθμόν); he does not know it *notionally* (kat¿ e@idos) of every triangle, even though there be actually no triangle which he does not know."

The difference between the phrase "used at one time to be proved" in the second illustration and "if any one should prove" in the third appears to indicate that, while the former referred to a historical fact, the latter does not; the reference ot a person who should prove the theorem of I. 32 for the three kinds of triangle separately, and then claim that he had proved it generally, states a purely hypothetical case, a mere illustration. Yet, coming after the historical fact stated in the preceding illustration, it might not unnaturally give the impression, at first sight, that it was historical too.

One the whole, therefore, it would seem that we cannot safely go behind the dictum of Eudemus that the discovery and proof of the theorem of I. 32 in all its generality were Pythagorean. This does not however preclude its having been discovered by stages such as those above set out after Hankel and Cantor. Nor need it be doubted that Thales and even his Egyptian instructors had advanced some way on the same road, so far at all events as to see that in an equilateral triangle, and in an isosceles right-angled triangle, the sum of the angles is equal to two right angles.

The Pythagorean proof.

This proof, handed down by Eudemus (Proclus, p. 379, 2–15), is no less elegant than that given by Euclid, and is a natural development from the last figure in the reconstructed argument of Hankel. It would be seen, after the theory of parallels was added to geometry, that the actual drawing of the perpendicular and the complete rectangle on BC as base was unnecessary, and that the parallel to BC through A was all that was required.



Let ABC be a triangle, and through A draw DE parallel to BC. [I. 31]

Then, since BC, DE are parallel,
the alternatve angles DAB, ABC are equal, [I. 29]
and so are the alternate angles EAC, ACB also.

Therefore the angles ABC, ACB are together equal to the angles DAB, EAC.

Add to each the angle BAC;
therefore the sum of the angles ABC, ACB, BAC is equal to the sum of the angles DAB, BAC, CAE, that is, to two right angles.

Euclid's proof pre-Euclidean.

The theorem of I. 32 is Aristotle's favourite illustration when he wishes to refer to some truth generally acknowledged and so often does it occur that it is often indicated by two or three words in themselves hardly intelligible, e.g. τὸ δυσὶν ὀρθαῖς (*Anal. Post.* I. 24, 85 b 5) and ὑπάρχει παντὶ τριγώνῳ τὸ δύο (*ibid.* 85 b 11).

One passage (*Metaph.* 1051 a 24) makes it clear as Heiberg (*op. cit.* p. 19) acutely observes, that in the proof as Aristotle knew it Euclid's construction was used. "Why does the triangle make up two right angles? Because *the angles about one point* are equal to two right angles. If then the parallel to the side had been *drawn up* (ἀνῆκτο), the fact would at once have been clear from merely looking at the figure." The words "the angles about one point" would equally fit the Pythagorean construction, but "drawn *upwards*" applied to the parallel to a side can only indicate Euclid's.

Attempts at proof independently of parallels.

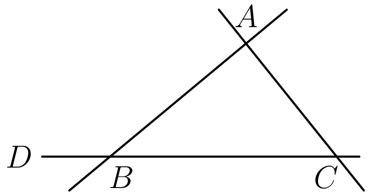
The most indefatigable worker on these lines was Legendre, and a sketch of his work has been given in the note on Postulate 5 above.

One other attempted proof needs to be mentioned here because it has found much favour. I allude to

Thibaut's method.

This appeared in Thibaut's *Grundriss der reinen Mathematik*, Göttingen (2 ed. 1809, 3 ed. 1818), and is to the following effect.

Suppose CB produced to D, and let BD (produced to any necessary extent either way) revolve in one direction (say clockwise) first about B into the position BA, then about A into the position of AC produced both ways, and lastly about C into the position CB produced both ways.



The argument then is that the straight line BD has revolved through the sum of the three exterior angles of the triangle. But, since it has at the end of the revolution assumed a position in the same straight line with its original position, it must have revolved *through four right angles*.

Therefore the sum of the three exterior angles is equal to four right angles; from which it follows that the sum of the three angles of the triangle is equal to two right angles.

But it is to be observed that the straight line BD revolves about *different points in it*, so that there is *translation* combined with *rotatory* motion, and it is necessary to assume as an axiom that the two motions are independent, and therefore the *translation* may be neglected.

Schumacher (letter to Gauss of 3 May, 1831) tried to represent the rotatory motion graphically in a second figure as mere motion round a point; but Gauss (letter of 17 May, 1831) pointed out in reply that he really assumed, without proving it, a proposition to the effect that "If two straight lines (1) and (2) which cut one another make angles A, A'' with a straight line (3) cutting both of them, and if a straight line (4) in the same plane is likewise cut by (1) at an angle A', then (4) will be cut by (2) at the angle A''. But this proposition not only needs proof, but we may say that it is, in *Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, p. 230).

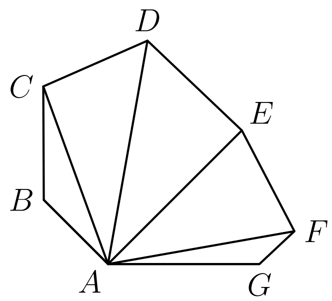
How easy it is to be deluded in this way is plainly shown by Proclus' attempt on the same lines. He says (p. 384, 13–21) that the truth of the theorem is borne in upon us by the help of "common notions" only. "For, if we conceive a straight line with two perpendiculars drawn to it at its extremities, and if we then suppose the perpendiculars to (revolve about their feet and) approach one another, so as to form a triangle, we see that, to the extent to which they converge, they diminish the right angles which they made with the straight ine, so that the amount taken from the right angles is also the amount added to the vertical angle of the triangle, and the three angles are necessarily made equal to two right angles." But a moment's reflection shows that, so far from being founded on mere "common notions," the supposed proof assumes, to being with, that, if the perpendiculars approach one another ever so litte, they will then form a triangle immediately, i.e., it assumes Postulate 5 itself; and the fact about the vertical angle can only be seen by means of the equality of the alternate angles exhibited by drawing a perpendicular from the vertex of the triangle to the base, i.e. a *parallel* to either of the original perpendiculars.

Extension to polygons.

The two important corollaries added to I. 32 in Simson's edition are given by Proclus; but Proclus' proof of the first is different from, and perhaps somewhat simpler than, Simson's.

1. *The sum of the interior angles of a convex rectilineal figure is equal to twice as many right angles as the figure has sides, less four.*

For let one angular point A be joined to all the other angular points with which it is not connected already.



The figure is then divided into triangles, and mere inspection shows

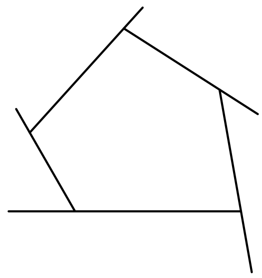
(1) that the number of triangles is tow less than the number of sides in the figure,

(2) that the sum of the angles of all the triangles is equal to the sum of all the interior angles of the figure.

Since then the sum of the angles of each triangle is equal to two right angles the sum of the interior angles of the figrue is equal to 2 (n - 2) right angles, i.e. (2n - 4) right angles, where n is the number of sides in the figure.

2. *The exterior angles of any convex rectilineal figure are together equal to four right angles.*

For the interior and exterior angles together are equal to $2n$ right angles, where n is the numebr of sides.



And the interior angles are together equal to $(2n - 4)$ right angles.

Therefore the exterior angles are together equal to four right angles.

This last property is already quoted by Aristotle as true of all rectilineal figures in two passages (*Anal. Post.* I. 24, 85 b 38 and II. 17, 99 a 19).