[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 312–314 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 29.]

23. **straight lines produced indefinitely from angles less than two right angles**, αἱ δὲ ἀπ᾽ ἐλασσόνων ἢ δύο ὀρθῶν ἐκβαλλόμεναι εἰς ἄπειρον συμπίπτουσιν, a variation from the more explicit language of Postulate 5. A good deal is left to be understood, namely that the straight lines begin from points at which they meet a transversal, and make with it internal angles on the same side the sum of which is less than two right angles.

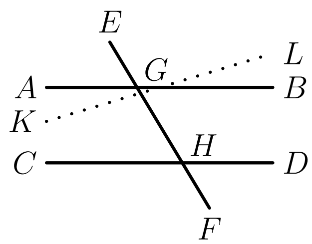
26. **because they are by hypothesis parallel**, literally "because they are supposed parallel," διὰ τὸ παραλλήλους αὐτὰς ὑποκεῖσθαι.

Proof by "Playfair's" axiom.

If, instead of Postulate 5, it is preferred to use "Playfair's" axiom in the proof of this proposition, we proceed thus.

To prove that the alternate angles AGH, GHD, are equal.

If they are not equal, draw another straight line KL through G making the angle KGH equal to the angle GHD.



Then, since the angles KGH, GHD are equal,

KL is parallel to CD. [I. 27]

Therefore *two straight lines* KL, AB*, intersecting at* G *are both parallel to the straight line* CD:

shape which is impossible (by the axiom).

Therefore the angle AGH cannot but be equal to the angle GHD.

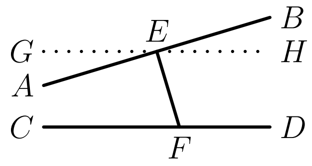
The rest of the proposition follows as in Euclid.

Proof of Euclid's Postulate 5 from "Playfair's" axiom.

Let AB, CD make with the transversal EF the angles AEF, EFC together less than two right angles.

To prove that AB, CD meet towards A, C.

Through E draw GH making with EF the angle GEF equal (and alternate) to the angle EFD.



Thus GH is parallel to CD. [I. 27]

Then (1) AB must meet CD in one direction or the other.

For, if it does not, AB must be parallel to CD; hence we have two straight lines AB, GH intersecting at E and both parallel to CD:
which is impossible.

Therefore AB, CD must meet.

(2) Since AB, CD meet, they must form a triangle with EF.

But in any triangle any two angles are together less than two right angles.

Therefore the angles AEF, EFC (which are less than two right angles), and not the angles BEF, EFD (which are together greater than two right angles, by I. 13), are the angles of the triangle;
that is, EA, FC meet in the direction of A, C, or on the side of EF on which are the angles together less than two right angles.

The usual course in modern text-books which use "Playfair's" axiom in lieu of Euclid's Postulate is apparently to prove I. 29 by means of the axiom, and then Euclid's Postulate by means of I. 29.

De Morgan would introduce the proof of Postulate 5 by means of "Playfair's" axiom *before* I. 29 and would therefore apparently prove I. 29 as Euclid does, without any change.

As between Euclid's Postulate 5 and "Playfair's" axiom, it would appear that the tendency in modern text-books is rather in favour of the latter. Thus, to take a few noteworthy foreign writers, we find that Rausenberger stands almost alone in using Euclid's Postulate, while Hilbert, Henrici and Treutlein, Rouché and De Comberousse, Enriques and Amaldi all use "Playfair's" axiom.

Yet the case for preferring Euclid's Postulate is argued with some force by Dodgson (*Euclid and his modern Rivals*, pp. 44–6). He maintains (1) that "Playfair's" axiom in fact involves Euclid's Postulate, but at the same time involves *more* than the latter, so that, to that extent, it is a needless strain on the faith of the learner. This is shown as follows.

Given AB, CD making with EF the angles AEF, EFC together less than two right angles, draw GH through E so that the angles GEF, EFC are together *equal* to two right angles.

Then, by I. 28 GH, CD are "separational."

We see then that any lines which have the property (α) that they make with a transversal angles less than two right angles have also the property (β) that one of them intersects a straight line which is "separational" from the other.

Now Playfair's axiom asserts that the lines which have the property (β) meet if produced: for, if they did not, we should have two intersecting straight lines both "separational" from a third, which is impossible.

We then argue that lines having property (α) meet because lines having property (α) are lines having property (β). But we do not know, until we have proved I. 29, that all pairs of lines having property (β) have also property (α). For anything we know to the contrary, class (β) *may* be greater than class (α). Hence if you assert anything of class (β), the logical effect is more extensive than if you assert it of class (α); for you assert it, not only of that portion of class (β) which is known to be included in class (α), but also of the unknown (but possibly existing) portion which is *not* so included.

(2) Euclid's Postulate puts before the beginner clear and *positive* conceptions, a pair of straight lines, a transversal, and two angles together less than two right angles, whereas "Playfair's" axiom requires him to realise a pair of straight lines which never meet though produced to infinity: a *negative* conception which does not convey to the mind any clear notion of the relative position of the lines. And (p. 68) Euclid's Postulate gives a direct criterion for judging that two straight lines meet, a criterion which is constantly required, e.g. in I. 44. It is true that the Postulate can be *deduced* from "Playfair's" axiom, but editors frequently omit to deduce it, and then tacitly assume it afterwards: which is the least justifiable course at all.