[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 308–309 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 27.]

1. **falling on two straight lines**, εἰς δύο εὐθειας ἐμπίπτουσα, the phrase being the same as that used in Post. 5, meaning a *transversal.*

2. **the alternate angles**, αἱ ἐναλλὰξ γωνίαι. Proclus (p. 357, 9) explains that Euclid uses the word *alternate* (or, more exactly, *alternately*, ἐναλλάξ) in two connexions, (1) of a certain transformation of a proportion, as in Book V. and the arithmetical Books, (2) as here, of certain of the angles formed by parallels with a straight line crossing them. *Alternate* angles are, according to Euclid as interpreted by Proclus, those which are not on the same side of the transversal, and are not adjacent but are separated by the transersal, both being within the parallels but one "above" and the other "below." The meaning is natural enough if we imaging the four internal angles to be taken in cyclic order and *alternate* to be any two of them not successive but separated by one angle of the four.

9. **in the direction B, D or towards A, C**, literally "towards the *parts* B, D or towards A, C", ἐπὶ τὰ Β, Δ μέρη ἢ ἐπὶ τὰ Α, Γ.

With this proposition begins the second section of the first Book. Up to this point Euclid has dealt mainly with triangles, their construction and their properties in the sense of the relation of their parts, the sides and angles, to one another, and the comparison of different triangles in respect of their parts, and of their area in the particular cases where they are congruent.

The second section leads up to the third, in which we pass to relations between the areas of triangles, parallelograms and squares, the special feature being a new conception of *equality* of areas, equality not dependent on *congruence*. This whole subject requires the use of parallels. Consequently the second section beginning at I. 27 establishes the theory of parallels, introduces the cognate matter of the equality of the sum of the angles of a triangle to two right angles (I. 32), and ends with two propositions forming the transition to the third section, namely I. 33, 34, which introduce the parallelogram for the first time.

Aristotle on parallels.

We have already seen reason to believe that Euclid's personal contribution to the subject was nothing less than the formulation of the famous Postulate 5 (see the notes on that Postulate and on Def. 23), since Aristotle indicates that the then current theory of parallels contained a *petitio principii*, and presumably therefore it was Euclid who saw the defect and proposed the remedy.

But it is clear that the propositions I. 27, 28 were contained in earlier text-books. They were familiar to Aristotle, as we may judge from two interesting passages.

(1) In *Anal. Post.* I. 5 he is explaining that a scientific demonstration must not only prove a fact of every individual of a class (κατὰ παντός) but must prove it primarily and generally true (πρῶτον καθόλου) of the *whole* of the class as one; it will not do to prove it first of one part, then of another part, and so on, until the class is exhausted. He illustrates this (74 a 13–16) by a reference to parallels: "If then one were to show that right (angles) do not meet, the proof of this might be thought to depend on the fact that this is true of all (pairs of actual) right angles. But this is not so, inasmuch as the result does not follow because (the angles are) equal (to two right angles) *in the particular way* [i.e. because each is a right angle], but by virtue of their being equal (to two right angles) in any way whatever [i.e. because the *sum* only needs to be equal to right angles, and the angles themselves may vary as much as we please subject to this]."

(2) The second passage has already been quoted in the note on Def. 23: "there is nothing surprising in different hypotheses leading to the same false conclusion; e.g., the conclusion that parallels meet might equally be drawn from either of the assumptions (*a*) that the interior (angle) is greater than the exterior or (*b*) that the sum of the angles of a triangle is greater than two right angles." (*Anal. Prior.* II. 17, 66 a 11–15).

I do not quite concur in the interpretation which Heiberg places upon these passages (*Mathematisches zu Aristoteles*, pp. 18–19). He says, first, that the allusion to the "interior angle" being "greater than the exterior" in the second passage shows that the reference in the first passage must be to Eucl. I 28 and not to I. 27, and he therefore takes the words ὅτι ὡδὶ ἴσαι in the first passage (which I have translated "because the two angles are equal to two right angles in the particular way") as meaning "because the angles, viz. the *exterior* and the *interior*, are equal in the particular way." He also takes αἱ ὀρθαὶ οὐ συμπίπτουσι (which I have translated "right angles do not meet," an expression quite in Aristotle's manner) to mean "perpendicular *straight lines* do not meet"; this is very awkward, especially as he is obliged to supply *angles* with ἴσαι in the next sentence.

But I think that the first passage certainly refers to I. 28, although I do not think that the alternative (*a*) in the second passage suggests it. This alternative may, I think, equally with the alternative (*b*) refer to I. 27. That proposition is proved by *reductio ad absurdum* based on the fact that, if the straight lines do meet, they must form a *triangle*, in which case the exterior angle must be greater than the interior (while according to the hypothesis these angles are equal). It is true that Aristotle speaks of the hypothesis that the *interior* angle is greater than the *exterior*; but after all Aristotle had only to state *some* incorrect hypothesis. It is of course only in connexion with straight lines *meeting*, as the hypothesis in I. 27 makes them, that the alternative (b) about the sum of the angles of a triangle could come in, and alternative (a) implies alternative (b).

It seems clear then from Aristotle that I. 27, 28 at least are pre-Euclidean, and that it was only in I. 29 that Euclid made a change by using his Postulate.

De Morgan observes that I. 27 is a *logical* equivalent of I. 16. Thus, if A means "straight lines forming a triangle with a transversal," B "straight lines making angles with a transversal on the same side which are together less than two right angles," we have

All A is B,

and it follows *logically* that

All not-B is not-A.