[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 303–307 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 26.]

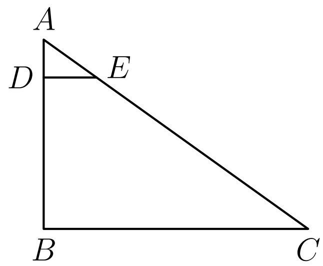
2–3. the **side adjoining the equal angles** πλευρὰν τὴν πρὸς ταῖς ἴσαις γωνίαις.

29. **is by hypothesis equal**. ὑπόκειται ἴση, according to the elegant Greek idiom. ὑπόκειμαι is used for the passive of ὑποτίθημι, as κεῖμαι is used for the passive of τίθημι, and so with the other compounds. Cf. προσκεῖσθαι, "to be added."

The alternative method of proving this proposition, viz. by applying one triangle to the other, was very early discovered, at least so far as regards the case where the equal sides are adjacent to the equal angles in each. An-Nairīzī gives it for this case, observing that the proof is one which he had found, but of which he did not know the author.

Proclus has the following interesting note (p. 352, 13–18): "Eudemus in his geometrical history refers this theorem to Thales. For he says that, in the method by which they say that Thales proved the distance of ships in the sea, it was necessary to make use of this theorem." As, unfortunately, this information is not sufficient of itself to enable us to determine how Thales solved this problem, there is considerable room for conjecture as to his method.

The suggestions of Bretschneider and Cantor agree in the assumption that the necessary observations were probably made from the top of some tower or structure of known height, and that a right-angled triangle was used in which the tower was the perpendicular, and the line connecting the bottom of the tower and the ship was the base, as in the annexed figure, where AB is the tower and C the ship. Bretschneider (*Die Geometrie und die Geometer vor



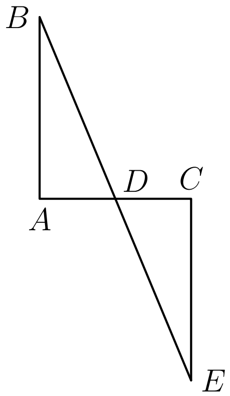
Eukleides*, § 30) says that it was only necessary for the observer to observe the angle CAB, and then the triangle would be completely determined by means of this angle and the known length AB. As Bretschneider says that the result would be obtained "in a moment" by this method, it is not clear in what sense he supposes Thales to have "observed" the angle BAC. Cantor is more definite (*Gesch. d. Math.* I$_3$, p. 145), for he says that the problem was nearly related to that of finding the *Seqt* from given sides. By the *Seqt* in the Papyrus Rhind is meant the ratio to one another of certain lines in pyramids or obelisks. Eisenlohr and Cantor took the one word to be equivalent, sometimes to the *cosine* of the angle made by the *edge* of the pyramid with the co-terminous diagonal of the base, sometimes to the *tangent* of the angle of slope of the *faces* of the pyramid. It is now certain that it meant one thing, viz. the ratio of half the side of the base to the height of the pyramid, i.e. the *cotangent* of the angle of slope. The calculation of the *Seqt* thus implying a sort of theory of similarity, or even of trigonometry, the suggestion of Cantor is apparently that the *Seqt* in this case would be found from a *small* right-angled triangle ADE having a common angle A with ABC as shown in the figure, and that the ascertained value of the *Seqt* with the length AB would determine BC. This amounts to the use of the property of similar triangles; and Bretschneider's suggestion must apparently come to the same thing, since, even if Thales measured the *angle* in our sense (e.g., by its ratio to a right angle), he would, in the absence of something corresponding to a table of trigonometrical ratios, have gained nothing and would have had to work out the proportions all the same.

Max C. P. Schmidt also (*Kulturhistorische Beiträge zur Kenntnis des griechischen and römischen Altertums*, 1906, p. 32) similarly supposes Thales to have had a right angle made of wood or bronze with the legs graduated, to have placed it in the position ADE (A being the position of his eye), and then to have read off the lengths AD, DE respectively, and worked out the length of BC by the rule of three.

How then does the supposed use of similar triangles and their property square with Eudemus' remark about I. 26? As it stands, it asserts the *equality* of *two* triangles which have two angles and one side respectively equal, and the theorem can only be brought into relation with the above explanations by taking it as asserting that, if two angles and one side of *one* triangle are given, the triangle is completely determined. But, if Thales practically used *proportions*, as supposed, I. 26 is surely not at all the theorem which this procedure would naturally suggest as underlying it and being "necessarily used"; the use of proportions or of similar but not equal triangles would surely have taken attention altogether away from I. 26 and fixed it on VI. 4.

For this reason I think Tannery is on the right road when he tries to find a solution using I. 26 as it stands, and withal as primitive as any recorded solution of such a problem. His suggestion (*La Géométrie grecque*, pp. 90-1) is based on the *fluminis varatio* of the Roman agrimensor Marcus Junius Nipsus and is as follows.

To find the distance from a point A to an inaccessible point B. From A measure along a straight line at right angles to AB a length AC and bisect it at D. From C draw CE at right angles to CA on the side of it remote from B, and let E be the point on it which is in a straight line with B and D.



Then, by I. 26, CE is obviously equal to AB.

As regards the equality of angles, it is to be observed that those at D are equal because they are vertically opposite, and, curiously enough, Thales is expressly credited with the discovery of the equality of such angles.

The only objection which I can see to Tannery's solution is that it would require, in the case of the ship, a certain extent of free and level ground for the construction and measurements.

I suggest therefore that the following may have been Thales' method. Assuming that he was on top of a tower, he had only to use a rough instrument made of a straight stick and a cross-piece fastened to it so as to be capable of turning about the fastening (say a nail) so that it could form any angle with the stick and would remain where it was put. Then the natural thing would be to fix the stick upright (by means of a plumb-line) and direct the cross-piece towards the ship. Next, leaving the cross-piece at the angle so found, the stick could be turned round, still remaining vertical, until the cross-piece pointed to some visible object on the shore, when the object could be mentally noted and the distance from the bottom of the tower to it could be subsequently measured. This would, by I. 26, give the distance from the bottom of the tower to the ship. This solution has the advantage of corresponding better to the simpler and more probable version of Thales' method of measuring the height of the pyramids; Diogenes Laertius says namely (I. 27, p. 6, ed. Cobet) on the authority of Hieronymus of Rhodes (B.C. 290–230), that he waited for this purpose until the moment when *our shadows are of the same length as ourselves.*

Recapitulation of congruence theorems.

Proclus, like other commentators, gives at this point (p. 347, 20 sqq.) a summary of the cases in which the equality of two triangles in all respects can be established. We may, he says, seek the conditions of such equality by successively considering as hypotheses the equality (1) of sides alone, (2) of angles alone, (3) of sides and angles combined. Taking (1) first, we can only establish the equality of the triangles in all respects if all three sides are respectively equal; we cannot establish the equality of the triangles by any hypothesis of class (2), not even the hypothesis that all the three angles are respectively equal; among the hypotheses of class (3), the equality of one side and one angle in each triangle is not enough, the equality (a) of one side and all three angles is more than enough, as is also the equality (b) of two sides and two or three angles, and (c) of three sides and one or two angles.

The only hypotheses therefore to be determined from this point of view are the equality of

(α) three sides [Eucl. I. 8].

(β) two sides and one angle [I. 4 proves one case of this, where the angle is that contained by the sides which are by hypothesis equal].

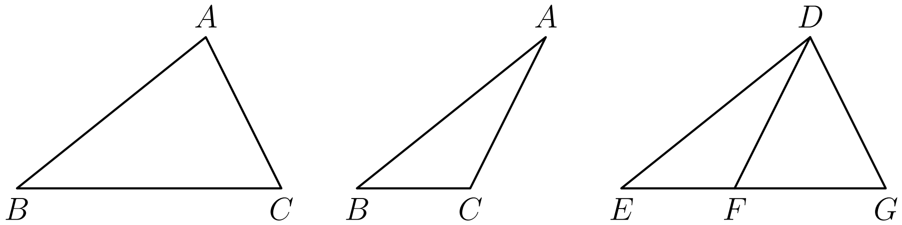
(γ) one side and two angles [I. 26 covers all cases].

It is curious that Proclus makes no allusion to what we call the *ambiguous case*, that case of (β) in which it is an angle opposite to one of the two specified sides in one triangle which is equal to the angle opposite to the equal side in the other triangle. Camerer indeed attributes to Proclus the observation that in this case the equality of the triangles cannot, unless some other condition is added, be asserted generally; but it would appear that Camerer was probably misled by a figure (Proclus, p. 351) which looks like a figure of the ambiguous case but is really used only used to show that in I. 26 the equal sides must be *corresponding* sides, i.e., they must be either adjacent to the equal angles in each triangle, or opposite to corresponding equal angles, and that, e.g., one of the equal sides must not be adjacent to the two angles in one triangle, while the side equal to it in the other triangle is opposite to one of the two corresponding angles in that triangle.

The ambiguous case.

If two triangles have two sides equal to two sides respectively, and if the angles opposite to one pair of equal sides be also equal, then will the angles opposite the other pair of equal sides be either equal or supplementary; and, in the former case, the triangles will be equal in all respects.

Let ABC, DEF be two triangles such that AB is equal to DE, and AC to DF, while the angle ABC is equal to the angle DEF;
it is required to prove that the angles ACB, DFE are either equal or supplementary.



Now (1), if the angle BAC be equal to the angle EDF, it follows, since the two sides AB, AC are equal to the two sides DE, DF respectively, that

the triangles ABC, DEF are equal in all respects. [I. 4]

and the angles ACB, DFE are equal.

(2) If the angles BAC, EDF be not equal, make the angle EDG (on the same side of ED as the angle EDF) equal to the angle BAC.

Let EF, produced if necessary, meet DG in G.

Then, in the triangles ABC, DEG,
the two angles BAC, ABC are equal to the two angles EDG, DEG respectively,
and the side AB is equal to the side DE;

therefore the angles ABC, DEG are equal in all respects, [I. 26]

so that the side AC is equal to the side DG,

and the angle ACB is equal to the angle DGE.

Again, since AC is equal to DF as well as to DG,

DF is equal to DG,

and therefore the angles DFG, DGF are equal.

But the angle DFE is supplementary to the angle DFG; and the angle DGF was proved equal to the angle ACB;

therefore the angle DFE is supplementary to the angle ACB.

If it is desired to avoid the ambiguity and secure that the triangles may be congruent, we can introduce the necessary conditions into the enunciation, on the analogy of Eucl. VI. 7.

If two triangles have two sides of the one equal to two sides of the other respectively, and the angles opposite to a pair of equal sides equal, then, if the angles opposite to the other pair of equal sides are both acute, or both obtuse, or if one of them is a right angle, the two triangles are equal in all respects.

The proof of the three cases (by *reductio ad absurdum*) was given by Todhunter.