[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 293–294 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 22.]

2–4. This is the first case in the *Elements* of a διορισμός to a problem in the sense of a statement of the conditions or limits of the possibility of a solution. The criterion is of course supplied by the preceding proposition.

2. **thus it is necessary.** This is usually translated (e.g., by Williamson and Simson) "*But* it is necessary," which is however inaccurate, since the Greek is not δεῖ δέ but δεῖ δή. The words are the same as those used ot introduce the διορισμός in the other sense of the "definition" or "particular statement" of a construction to be effected. Hence, as in the latter case we say "thus it is required" (e.g., to bisect the finite straight line AB, I. 10), we should here translate "*thus* it is necessary."

4. To this enunciation all the MSS. and Boethius add, after the διορισμός, the words "because in any triangle two sides taken together in any manner are greater than the remaining one." But this explanation has the appearance of a gloss, and it is omitted by Proclus and Campanus. Moreover there is no corresponding addition to the διορισμός of VI. 28.

It was early observed that Euclid assumes, without giving any reason, that the circles drawn as described will meet if the condition that any two of the straight lines A, B, C are together greater than the third be fulfilled. Proclus (p. 331, 8 sqq.) argues the matter by means of a *reductio ad absurdum*, but does not exhaust the possible hypotheses inconsistent with the contention. He says the circles must do one of three things, (1) cut one another, (2) touch one another, (3) stand apart (διεστάναι) from one another. He then considers the hypotheses (a) of their touching *externally*, (b) of their being separated from one another by a space. He should have considered the hypothesis (c) of one circle touching the other *internally* or lying entirely within the other without touching. These three hypotheses being successively disproved, it follows that the circles must meet (this is the line taken by Camerer and Todhunter).

Simson says: "Some authors blame Euclid because he does not demonstrate that the two circles made use of in the construction of this problem must cut one another: but this is very plain from the determination he has given, namely, that any two of the straight lines DF, FG, GH must be greater than the third. For who is so dull, though only beginning to learn the Elements, as not to perceive that the circle described from the centre F, at the distance FD, must meet FH betwixt F and H, because FD is less than FH; and that, for the like reason, the circle described from the centre G at the distance GH must meet DG betwixt D and G; and that these circles must meet one another, because FD and GH are together greater than FG."

We have in fact only to satisfy ourselves that one of the circles e.g., that with centre G has at least one point of its circumference inside the same circle; and this is best shown with reference to the points in which the first circle cuts the straight line DE. For (1) FH, being equal to the sum of B and C, is greater than A, i.e. than the radius of the circle with centre F, and therefore H is outside the circle. (2) If GM be measured along GF equal to GH or C, then, since GM is either (a) less or (b) greater than GF, M will fall (a) between G and F or (b) beyond F towards D; in the first case (a) the sum of FM and C is equal to FG and therefore less than the sum of A and C, so that FM is less than A or FD; in the second case (b) the sum of MF and FG i.e. the sum of MF and B, is equal to GM or C, and therefore less than the sum of A and B, so that MF is less than A or FD; hence in either case M falls within the circle with centre F.

It being now proved that the circumference of the circle with centre G has at least one point outside, and at least one point inside, the circle with centre F, we have only to invoke the Principle of Continuity, as we have to do in I. 1 (cf. the note on that proposition, p. 242, where the necessary postulate is stated in the form suggested by Killing).

That the construction of the proposition gives only *two* points of intersection between the circles, and therefore only two triangles satisfying the condition, one on each side of FG, is clear from I. 7, which, as before pointed out, takes the place, in Book I., of III. 10 proving that two circles cannot intersect in more points than two.