[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 289–292 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 21.]

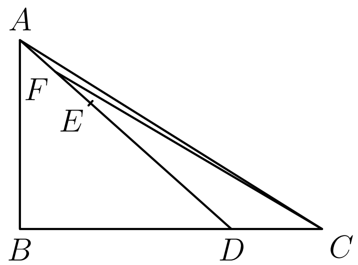
2. **be constructed ... meeting within the triangle.** The word "meeting" is not in the Greek, where the words are ἐντὸς συσταθῶσιν. συνίστασθαι is the word used of constructing two straight lines *to a point* (cf. I. 7) or so as to form a triangle; but it is necessary in English to indicated that they *meet*.

3. **the straight lines so constructed.** Observe the elegant brevity of the Greek αἱ συσταθεῖσαι.

The editors generally call attention to the fact that the lines drawn within the triangle in this proposition must be drawn, as the enunciation says, from the *ends* of the side; otherwise it is not necessary that their sum should be less than that of the remaining sides of the triangle. Proclus (p. 327, 12 sqq.) gives a simple illustration.

Let ABC be a right-angled triangle. Take any point D on BC, join DA, and cut off from it DE equal to AB. Bisect AE at F, and join FC.

Then shall CF, FD be together greater than CA, AB.



For CF, FE are equal to CF, FA,
and therefore greater than CA.
Add the equals ED, AB respectively;
therefore CF, FD are together greater than CA, AB.

Pappus gives the same proposition as that just proved, but follows it up by a number of others more elaborate in character, selected apparently from "the so-called paradoxes" of one Erycinus (Pappus, III. p. 106 sqq.). Thus he proves the following:

1. In any triangle, except an equilateral triangle or an isosceles triangle with base less than one of the other sides, it is possible to construct on the base and within the triangle two straight lines the sum of which is equal to the sum of the other two sides of the triangle.

2. In any triangle in which it is possible to construct two straight lines on the base which are equal to the sum of the other two sides of the triangle it is also possible to construct two others the sum of which is *greater* than that sum.

3. Under the same conditions, if the base is greater than either of the other two sides, two straight lines can be constructed in the manner described which are *respectively* greater than the other two sides of the triangle; and the lines may be constructed so as to be respectively *equal* to the two sides, if one of those two sides is less than the other and each of them less than the base.

4. The lines may be so constructed that their sum will bear to the sum of the two sides of the triangle any ratio less than 2 : 1.

As a specimen of the proofs we will give that of the proposition which has been numbered (1) for the case where the triangle is isosceles (Pappus, III. pp. 108–110).

Let ABC be an isosceles triangle in which the base AC is greater than either of the equal sides AB, BC.

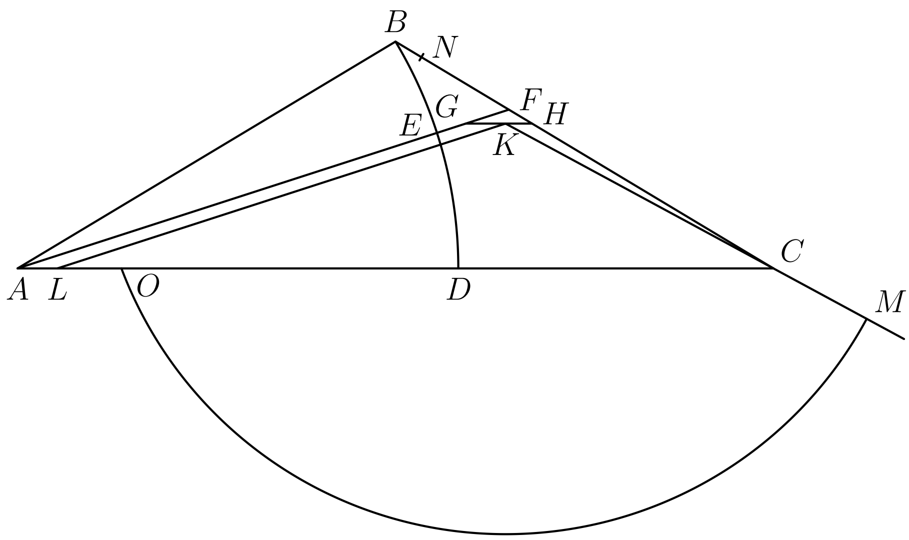
With centre A and radius AB describe a circle meeting AC in D.

Draw any radius AEF such that it meets BC in a point F outside the circle.

Take any point G on EF, and through it draw GH parallel to AC. Take any point K on GH, and draw KL parallel to FA meeting AC in L.

From BC cut off BN equal to EG.

Thus AG, or LK, is equal to the sum of AB, BN, and CN is less than LK.



Now GF, FH are together greater than GH,
and CH, HK together greater than CK.

Therefore, by addition,
CF, FG, HK are together greater than CK, HG.

Subtracting HK from each side, we see that
CF, FG are together greater than CK, KG;
therefore, if we add AG to each,
AF, FC are together greater than AG, GK, KC.

And AB, BC are together greater than AF, FC. [I. 21]

Therefore AB, BC are together greater than AG, GK, KC.

But, by construction, AB, BN are together equal to AG;
therefore, by subtraction, NC is greater than GK, KC,
and *a fortiori* greater than KC.

Take on KC produced a point M such that KM is equal to NC; with centre K and radius KM describe a circle meeting CL in O, and join KO.

Then shall LK, KO be equal to AB, BC.

For, by construction, LK is equal to the sum of AB, BN, and KO is equal to NC;
therefore LK, KO are together equal to AB, BC.

It is after I. 21 that (as remarked by De Morgan) the important proposition about the perpendicular and obliques drawn from a point to a straight line of unlimited length is best introduced:

Of all straight lines that can be drawn to a given straight line of unlimited length from a given point without it:

(a) *the perpendicular is the shortest;*

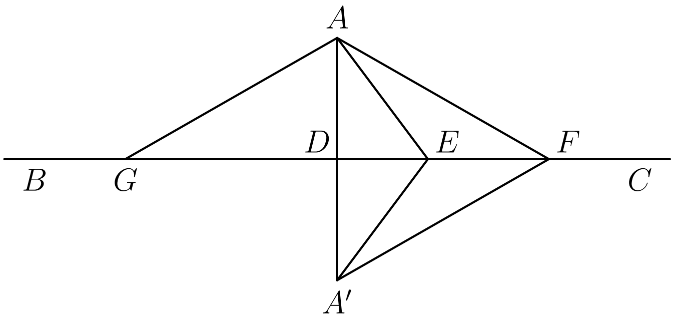
(b) *of the obliques, that is greater the foot of which is further from the perpendicular;*

(c) *given one oblique, only one other can be found of the same length, namely that the foot of which is equally distant with the foot of the given one from the perpendicular, but on the other side of it.*

Let A be the given point, BC the given straight line; let AD be the perpendicular from A on BC, and AE, AF any two obliques of which AF makes the greater angle with AD.

Produce AD to A', making $A'D$ equal to AD, and join $A'E$, $A'F$.

Then the triangles ADE, $A'DE$ are equal in all respects; and so are the triangles ADF, $A'DF$.



Now (1) in the triangle AEA' the two sides AE, EA' are greater than AA' [I. 20], that is, twice AE is greater than twice AD.

Therefore AE is greater than AD.

(2) Since AE, $A'E$ are drawn to E, a point within the triangle AFA',

AF, FA' are together greater than AE, EA', [I. 21]

or twice AF is greater than twice AE.

Therefore AF is greater than AE.

(3) Along DB measure off DG equal to DF, and join AG.

The triangles AGD, AFD are then equal in all respects, so that the angles GAD, FAD are equal, and AG is equal to AF.