[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 284–286 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 19.]

This proposition, like I. 6, can be proved by merely *logical* deduction from I. 5 and I. 18 taken together, as pointed out by De Morgan. The general form of the argument used by De Morgan is given in his *Formal Logic* (1847), p. 25, thus:

"*Hypothesis.* Let there be any number of propositions or assertions—three for instance, X, Y and Z—of which it is the property that one or the other must be true, *and one only.* Let there be three other propositions P, Q and R of which it is also the property that one, and one only, must be true. Let it be a connexion of those assertions that:

when X is true, P is true,
when Y is true, Q is true,
when Z is true, R is true.

Consequence: then it follows that,

when P is true, X is true,
when Q is true, Y is true,
when R is true, Z is true."

To apply this to the case before us, let us denote the sides of the triangle ABC by a, b, c, and the angles opposite to these sides by A, B, C respectively, and suppose that a is the base.

Then we have the three propositions,

$$\begin{array}{ll} \text{when } b \text{ is equal to } c,\ B \text{ is equal to } C, & [\text{I. 5}] \\ \left.\begin{array}{l} \text{when } b \text{ is greater than } c,\ B \text{ is greater than } C, \\ \text{when } b \text{ is less than } c,\ B \text{ is less than } C, \end{array}\right\} & [\text{I. 18}] \end{array}$$

and it follows *logically* that,

$$\begin{array}{ll} \text{when } B \text{ is equal to } C,\ b \text{ is equal to } c, & [\text{I. 6}] \\ \left.\begin{array}{l} \text{when } B \text{ is greater than } C,\ b \text{ is greater than } c, \\ \text{when } B \text{ is less than } C,\ b \text{ is less than } c. \end{array}\right\} & [\text{I. 19}] \end{array}$$

Reductio ad absurdum by exhaustion.

Here, says Proclus (p. 318, 16–23), Euclid proves the impossibility "by means of *division*" (ἐκ διαιρέσεως). This means simply the separation of different hypotheses, each of which is inconsistent with the truth of the theorem to be proved, and which therefore must be successively shown to be impossible. If a straight line is not greater than a straight line, it must be either equal to it or less; thus in a *reductio ad absurdum* intended to prove such a theorem as I. 19 it is necessary to dispose successively of *two* hypotheses inconsistent with the truth of the theorem.

Alternative (direct) proof.

Proclus gives a direct proof (pp. 319–321) which an-Nairīzı also has and attributes to Heron. It requires a lemma and is consequently open to the slight objection of separating a theorem from its converse. But the lemma and proof are worth giving.

Lemma.

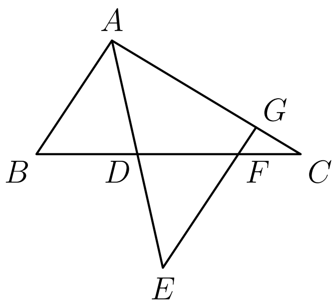
If an angle of a triangle be bisected and the straight line bisecting it meet the base and divide it into unequal parts, the sides containing the angle will be unequal, and the greater will be that which meets the greater segment of the base, and the less that which meets the less.

Let AD, the bisector of the angle A of the triangle ABC, meet BC in D, making CD greater than BD.

I say that AC is greater than AB.

Produce AD to E so that DE is equal to AD. And, since DC is greater than BD, cut off DF equal to BD.

Join EF and produce it to G.



Then, since the two sides AD, DB are equal to the two sides ED, DF, and the vertical angles at D are equal,

AB is equal to EF,

and the angle DEF to the angle BAD,

i.e. to the angle DAG (by hypothesis).

Therefore AG is equal to EG,

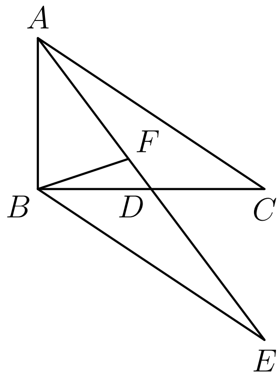
and therefore greater than EF, or AB.

Hence, *a fortiori*, AC is greater than AB.

Proof of I. 19.

Let ABC be a triangle in which the angle ABC is greater than the angle ACB.

Bisect BC at D, join AD, and produce it to E so that DE is equal to AD. Join BE.



Then the two sides BD, DE are equal to the two sides CD, DA, and the vertical angles at D are equal;

therefore BE is equal to AC,

and the angle DBE to the angle at C.

But the angle at C is less than the angle ABC;

therefore the angle DBE is less than the angle ABD.

Hence, if BF bisect the angle, ABE, BF meets AE between A and D. Therefore EF is greater than FA.

It follows, by the lemma, that BE is greater than BA,

that is, AC is greater than AB.