[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 280–281 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 16.]

1. **the exterior angle**, literally "the outside angle," ἡ ἐκτὸς γωνία.

2. **the interior and opposite angles**, τῶν ἐντὸς καὶ ἀπεναντίον γωνιῶν.

12. **let AC be drawn through to G**. The word is διήχθω, a variation on the more usual ἐκβεβλήσθω, "let it be *produced*."

21. **CFE**, in the text "FEC."

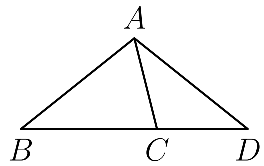
As is well known, this proposition is not universally true under the Riemann hypothesis of a space endless in extent but not infinite in size. On this hypothesis a straight line is a "closed series" and returns on itself; and two straight lines which have one point of intersection have another point of intersection also, which bisects the whole length of the straight line measured from the first point on it to the same point again; thus the axiom of Euclidean geometry that two straight lines do not enclose a space does not hold. If 4Δ denotes the finite length of a straight line measured from any point once round to the same point again, 2Δ is the distance between any two intersections of two straight lines which meet. Two points A, B do not determine one sole straight line unless the distance between them is different from 2Δ. In order that there may only be one perpendicular from a point C to a straight line AB, C must not be one of the two "poles" of the straight line.

Now, in order that the proof of the present proposition may be universally valid, it is necessary that CF should always fall within the angle ACD so that the angle ACF may be less than the angle ACD. But this will not always be so on the Riemann hypothesis. For, (1) if BE is equal to Δ, so that BF is equal to 2Δ, F will be the second point in which BE and BD intersect; i.e. F will lie on CD, and the angle ACF will be *equal* to the angle ACD. In this case the exterior angle ACD will be *equal* to the interior angle BAC. (2) If BE is greater than Δ and less than 2Δ, so that BF is greater than 2Δ and less than 4Δ, the angle ACF will be *greater* than the angle ACD, and therefore the angle ACD will be *less* than the interior angle BAC. Thus, e.g., in the particular case of a right-angled triangle, the angles other than the right angle may be (1) both acute, (2) one acute and one obtuse, or (3) both obtuse according as the perpendicular sides are (1) both less than Δ, (2) one less and the other greater than Δ, (3) both greater than Δ.

Proclus tells us (p. 307, 1–12) that some combined this theorem with the next in one enunciation thus: *In any triangle, if one side be produced, the exterior angle of the triangle is greater than either of the interior and opposite angles, and any two of the interior angles are less than two right angles*, the combination having been suggested by the similar enunciation of Euclid I. 32, *In any triangle, if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of the triangle are equal to two right angles.*

The present proposition enables Proclus to prove what he did not succeed in establishing conclusively in his note on I. 12, namely that *f*rom one point there cannot be drawn to the same straight line three straight lines equal in length.

For, if possible, let AB, AC, AD be all equal, B, C and D being in a straight line.



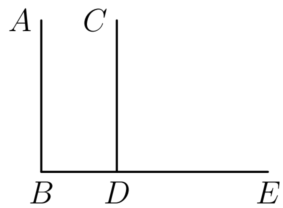
Then, since AB, AC are equal, the angles ABC, ACB are equal.

Similarly, since AB, AD are equal, the angles ABD, ADB are equal.

Therefore the angle ACB is equal to the angle ADC, i.e. the exterior angle to the interior and opposite angle: which is impossible.

Proclus next (p. 308, 14 sqq.) undertakes to prove by means of I. 16 that, *if a straight line falling on two straight lines make the exterior angle equal to the interior and opposite angle, the two straight lines will not form a triangle or meet*, for in that case the same angle would be both greater and equal.

The proof is really equivalent to that of Eucl. 27. If BE falls on the two straight lines AB, CD in such a way that the angle CDE is equal to the interior and opposite angle ABD, AB and CD cannot form a triangle or meet. For, if they did, then (by I. 16) the angle CDE would be *greater* than the angle ABD, while by the hypothesis it is at the same time *equal* to it.



Hence, says Proclus, in order that BA, DC may form a triangle it is necessary for them to *approach* one another in the sense of being turned round one pair of corresponding extremities, e.g. B, D, so that the other extremities A, C come nearer. This may be brought about in one of three ways (1) AB may remain fixed and CD be turned about D so that the angle CDE increases; (2) CD may remain fixed and AB be turned about B so that the angle ABD becomes smaller; (3) both AB and CD may move so as to make the angle ABD smaller and the angle CDE larger at the same time. The *reason*, then, of the straight lines AB, CD coming to form a triangle or to meet is (says Proclus) *the movement of the straight lines*.

Though he does not mention it here, Proclus does in another passage (p. 371, 2–10, quoted on p. 207 above) hint at the possibility that, while I. 16 may remain universally true, either of the straight lines BA, DC (or both together) may be turned through any angle not greater than a certain finite angle and yet may not meet (the Bolyai-Lobachewsky hypothesis).