[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 271–275 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 12.]

2. **a perpendicular straight line**, κάθετον εὐθεῖαν γραμμήν. This is the full expression for a *perpendicular*, κάθετος meaning *let fall* or *let down*, so that the expression corresponds to our *plumb-line*. ἡ κάθετος is however constantly used alone for a perpendicular, γραμμή being understood.

10. **on the other side of the straight line AB**, literally "towards the other parts of the straight line AB," ἐπὶ τὰ ἕτερα μέρη τῆς AB. Cf. "on the same side" (ἐπὶ τὰ αὐτὰ μέρη) in Post. 5 and "in both directions" (ἐφ᾽ ἑκάτερα τὰ μέρη) in Def. 23.

"This problem," says Proclus (p. 283, 7–10), "was first investigated by Oenopides [5th cent. B.C.], who thought it useful for astronomy. He however calls the perpendicular, in the archaic manner, (a line drawn) *gnomon-wise* (κατὰ γνώμονα), because the gnomon is also at right angles to the horizon." In this earlier sense the *gnomon* was a staff placed in a vertical position for the purpose of casting shadows and so serving as a means of measuring time (Cantor, *Geschichte der Mathematik*, I$_3$, p. 161). The later meanings of the word as used in Eucl. Book II. and elsewhere will be explained in the note on Book II. Def. 2.

Proclus says that two kinds of perpendicular were distinguished, the "plane" (ἐπίπεδος) and the "solid" (στερεά), the former being the perpendicular dropped on a line in a plane and the latter the perpendicular dropped on a plane. The term "solid perpendicular" is sufficiently curious, but it may perhaps be compared with the Greek term "solid locus" applied to a conic section, apparently on the ground that it has it origin in the second of a solid, namely a cone.

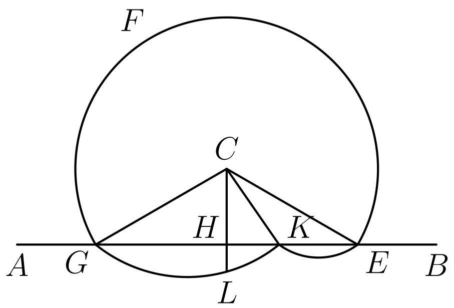
Attention is called by most editors to the assumption in this proposition that, if only D be taken on the side of AB remote from C, the circle described with CD as radius must necessarily cut AB in two points. To satisfy us of this we need, as in I. 1, some postulate of continuity, e.g. something like that suggested by Killing (see note on the Principle of Continuity above, p 235): "If a point [here the point describing the circle] moves in a figure which is divided into two parts [by the straight line], and if it belongs at the beginning of the motion to one part and at another stage of the motion to the other part, it must during the motion cut the boundary between the two parts," and this of course applies to the motion in *two* directions from D.

But the editors have not, as a rule, noticed a possible *objection* to the Euclidean statement of this problem which is much more difficult to dispose of at this stage, i.e. without employing any proposition later than this in

Euclid's order. How do we know, says the supposed critic, that the circle does not cut AB in *three* or more points, in which cae there would be not *one* perpendicular but *three* or more? Proclus (pp. 286, 12–289, 6) tries to refute this objection, and it is interesting to follow his argument, though it will easily be seen to be inconclusive. He takes in order three possible suppositions.

1. May not the circle meet AB in a third point K between the middle point of GE and either extremity of it, taking the form drawn in the figure appended?

Suppose this possible. Bisect GE in H. Join CH, and produce it to meet the circle in L. Join CG, CK and CE.



Then, since CG is equal to CE, and CH is common, while the base GH is equal to the base HE,

the angles CHG, CHE are equal and, since they are adjacent, they are both right.

Again, since CG is equal to CE,

the angles at G and E are equal.

Lastly, since CK is equal to CG and also to CE, the angles CGK, CKG are equal, as also are the angles CKE, CEK.

Since the angles CGK, CEK are equal, it follows that

the angles CKG, CKE are equal and therefore both right.

Therefore the angle CKH is equal to the angle CHK,

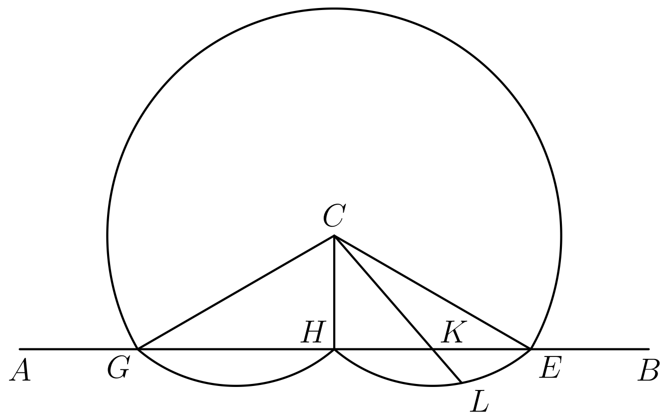
and CH is equal to CK.

But CK is equal to CL, by the definition of the circle; therefore CH is equal to CL: which is impossible.

Thus Proclus; but why should not the circle meet AB in H as well as K?

May not the circle meet AB in H the middle point of GE and take the form shown in the second figure?

In that case, says Proclus, join CG, CH, CE as before. Then bisect HE at K, join CK and produce it to meet the circumference at L.



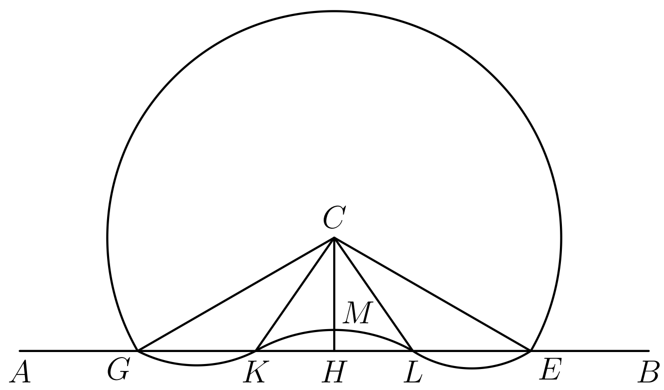
Now, since HK is equal to KE, CK is common, and the base CH is equal to the base CE,

the angles at K are equal and therefore both right angles.

Therefore the angle CHK is equal to the angle CKH, whence CK is equal to CH and therefore to CL: which is impossible.

So Proclus; but why should not the circle meet AB in K as well as H?

3. May not the circle meet AB in *two* points besides G, E and pass, between those two points, to the side of AB towards C, as in the next figure?



Here again, by the same method, Proclus proves that, K, L being the other two points in which the circle cuts AB,

CK is equal to CH,

and, since the circle cuts CH in M,

CM is equal to CK and therefore to CH: which is impossible.

But again, why should the circle not cut AB in the point H as well?

In fact, Proclus' cases are not mutually exclusive, and his method of proof only enables us to show that, if the circle meets AB in one more point besides G, E, it must meet it in more points still. We can always find a new point of intersection by bisecting the distance separating any two points of intersection, and so, applying the method *ad infinitum*, we should have to conclude ultimately that the circle with radius CH (or CG) coincides with AB. It would follow that a circle with centre C and radius greater than CH would not meet AB at all. Also, since all straight lines from C to points on AB would be equal in length, there would be an infinite number of perpendiculars from C on AB.

Is this under any circumstances possible? It is not possible in Euclidean space, but it is possible, under the Riemann hypothesis (where a straight line is a "closed series" and returns on itself), in the case where C is the pole of the straight line AB.

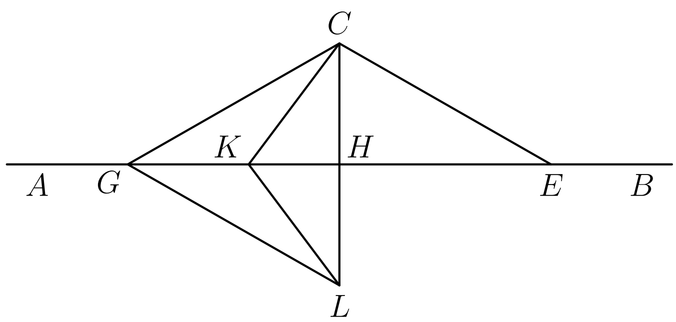
It is natural therefore that, for a proof that in Euclidean space there is only one perpendicular from a point to a straight line, we have to wait until I. 16, the precise proposition which under the Riemann hypothesis is only valid with a certain restriction and not universally. There is no difficulty involved by waiting until I. 16, since I. 12 is not used before that proposition is reached; and we are only in the same position as when, in order to satisfy ourselves of the number of possible solutions of I. 1, we have to wait until I. 7.

But if we wish, after all, to prove the truth of the assumption *without* recourse to any later proposition than I. 12, we can do so by means of this same invaluable I. 7.

If the circle intersects AB as before in G, E, let H be the middle point of GE, and suppose, if possible, that the circle also intersects AB in any other point K on AH.

From K, on the side of AB opposite to C, draw HL at right angles to AB, and make HL equal to HC.

Join CG, LG, CK, LK.



Now, in the triangles CHG, LHG, CH is equal to LH, and HG is common.

Also the angles CHG, LHG, being both right, are equal.

Therefore the base CG is equal to the base LG.

Similarly we prove that CK is equal to LK.

But, by hypothesis, since K is on the circle,

CK is equal to CG.

Therefore CG, CK, LG, LK are all equal.

Now the next proposition, I. 13 will tell us that CH, HL are in a straight line; but we will not assume this. Join CL.

Then on the same base CL and on the same side of it we have two pairs of straight lines drawn from C, L to G and K such that CG is equal to CK and LG to LK.

But this is impossible [I. 7].

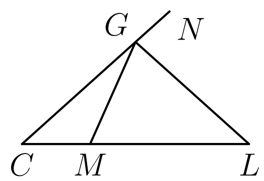
Therefore the circle cannot cut BA or BA produced in any point other than G on that side of CL on which G is.

Similarly it cannot cut AB or AB produced at any point other than E on the other side of CL.

The only possibility left therefore is that that the circle might cut AB in the same point as that in which CL cuts it. But this is shown to be impossible by an adaption of the proof of I. 7.

For the assumption is that there may be some point M on CL such that CM is equal to CG and LM to LG.

If possible, let this be the case, and produce CG to N.



Then, since CM is equal to CG,
the angle NGM is equal to the angle GML [I. 5, part 2].

Therefore the angle GML is greater than the angle MGL.

Again, since LG is equal to LM, the angle GML is equal to the angle MGL.

But it was also greater: which is impossible.

Hence the circle in the original figure cannot cut AB in the point in which CL cuts it.

Therefore the circle cannot cut AB in any point whatever except G and E.

[This proof of course does not prove that CK is *less* than CG, but only that it is not equal to it. The proposition that, of the obliques drawn from C

to AB, that is less the foot of which is nearer to H can only be proved later. The proof of I. 7 also fails, under the Riemann hypothesis, if C, L are the poles of the straight line B, since the broken lines CGL, CKL etc. become equal straight lines, all perpendicular to AB.]

Proclus rightly adds (p. 289, 18 sqq.) that it is not *necessary* to take D on the side of AB away from A if an objector "says that there is no space on that side." If it is not desired to trespass on that side of AB, we can take D anywhere on AB and described the *arc* of a circle between D and the point where it meets AB again, drawing the arc on the side of AB on which C is. If it should happen that the selected point D is such that the circle only meets AB in *one* point (D itself), we have only to describe the circle with CD as radius, then, if E be a point on this circle, take F a point further from C than E is, and describe with CF as radius the circular arc meeting AB in two points.