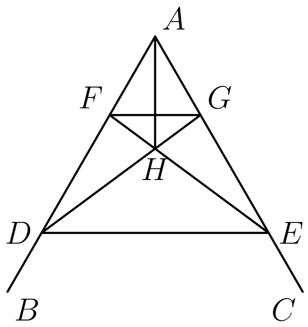
[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 264–267 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 9.]

It will be observed from the translation of this proposition that Euclid does not say, in his description of the construction, that the equilateral triangle should be constructed on the side of DE opposite to A; he leaves this to be inferred from his figure. There is no particular value in Proclus' explanation as to how we should proceed in case any one should assert that he could not recognise the existence of any space below DE. He supposes, then, the equilateral triangle described on the side of DE towards A, and hence has to consider three cases according as the vertex of the equilateral triangle falls on A, above A or below it. The second an third cases do not differ substantially from Euclid's. In the first case, where ADE is the equilateral triangle constructed on DE, take any point F on AD, and from AE cut off AG equal to AF. Join DG, EF meeting in H; and join AH. Then AH is the bisector required.



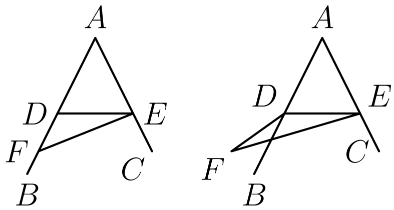
Proclus also answers the possible *objection* that might be raised to Euclid's proof on the ground that it assumes that, if the equilateral triangle be described on the side of DE opposite to A, its vertex F will lie within the angle BAC. The objector is supposed to argue that this is not necessary, but that F might fall either on one of the lines forming the angle or outside it altogether. The two cases are disposed of thus.

Suppose F to fall as shown in the two figures below respectively.

Then, since FD is equal to FE,
the angle FDE is equal to the angle FED.

Therefore the angle CED is greater than the angle FDE; and, in the second figure, *a fortiori*, the angle CED is greater than the angle BDE.

But, since ADE is an isosceles triangle, and the equal sides are produced, the angles under the base are equal,
i.e., the angle CED is equal to the angle BDE.



But the angle CED was proved greater: which is impossible.

Here then is the second case in which, in Proclus's view, the second part of I. 5 is useful for refuting objections.

On this proposition Proclus takes occasion (p. 271, 15–19) to emphasize the fact that the given angle must be *rectilineal*, since the bisection of any sort of angle (including angles made by curves with one another or with straight lines) is not matter for an elementary treatise, besides which it is questionable whether such bisection is always possible. "Thus it is difficult to say whether it is possible to bisect the so-called *horn-like* angle" (formed by the circumference of a circle and a tangent to it).

Trisection of an angle.

Further it is here that Proclus gives us his valuable historical note about the *trisection* of any acute angle, which (as well as the division of an angle in any given ratio) requires resort to other curves than circles, i.e. curves of the species which, after Geminus, he calls "mixed." "This," he says (p. 272, 1–12), "is shown by those who have set themselves the task of trisecting such a given rectilineal angle. For Nicomedes trisected any rectilineal angle by means of the *conchoidal* lines, the origin, order, and properties of which he has handed down to us, being himself the discoverer of their peculiarity. Others have done the same thing by means of the *quadratrices* of Hippias and Nicomedes, thereby again using 'mixed' curves. But others, starting from the Archimedean spirals, cut a given rectilineal angle in a given ratio."

(a) Trisection by means of the *conchoid*.

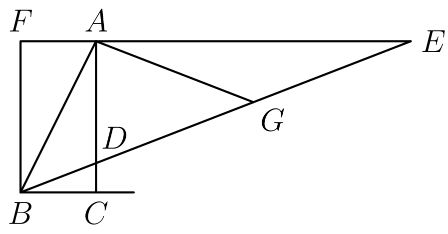
I have already spoken of the *conchoid* of Nicomedes (note on Def. 2, pp. 160–1)[1]; it remains to show how it could be used for trisecting an angle. Pappus explains this (IV. pp. 274–5) as follows.

[1] [**Footnote added by DRW:** Heath's note referred to here reads as follows: "As the only other curve, besides the parabola and the hyperbola, which has been mentioned as proceeding to infinity is the *conchoid* (of Nicomedes), we can hardly avoid the conclusion of Tannery that the curve which has a loop and then proceeds to infinity is a variety of the *conchoid* itself. As is well known, the ordinary conchoid (which was used both for doubling the cube and for trisecting the angle) is obtained in this way. Suppose any number of rays passing through a fixed point (the *pole*) and intersecting a fixed straight line; and suppose that points are taken on the rays, beyond the fixed straight line, such that the portions of

Let ABC be the given acute angle, and from any point A in AB draw AC perpendicular to BC.

Complete the parallelogram $FBCA$ and produce FA to a point E such that, if BE be joined, *BE intercepts between AC and AE a length DE equal to twice AB.*

I say that the angle EBC is one-third of the angle ABC.



For joining A to G the middle point of DE, we have the three straight lines AG, DG, EG equal, and the angle AGD is double of the angle AED or EBC.

But DE is double of AB;
therefore AG, which is equal to DG, is equal to AB.

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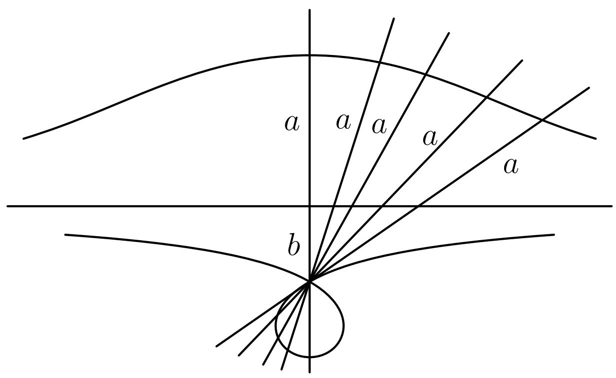
the rays intercepted between the fixed straight line and the point are equal to a constant *distance* (διάστημα), the locus of the points is a conchoid which has a fixed straight line for asymptote.



If the "distance" a is measured from the intersection of the ray with the given straight line, not in the direction away from the pole, but towards the pole, we obtain three other curves according as a is less than, equal to, or greater than b, the distance of the pole from the fixed straight line, which is an asymptote in each case. The case in which $a > b$ gives a curve which forms a loop and then proceeds to infinity in the way Proclus describes. Now we know both from Eutocius (*Comm. on Archimedes*, ed. Heiberg, III. p. 98) and Proclus (p. 272, 3–7) that Nicomedes wrote on conchoides (in the plural), and Pappus (IV. p. 244, 18) says that besides the "first" (used as above stated) there were "the second, the third and the fourth which are useful for other theorems."]

Hence the angle AGD is equal to the angle ABG.

Therefore the angle ABD is also double of the angle EBC; so that the angle EBC is one-third of the angle ABC.

So far Pappus, who reduces the construction to the drawing of BE so that DE shall be equal to twice AB.

This is what the conchoid constructed with B as *pole*, AC as *directrix*, and *distance* equal to twice AB enables us to do; for that conchoid cuts AE in the required point E.

(b) Use of the *quadratrix*.

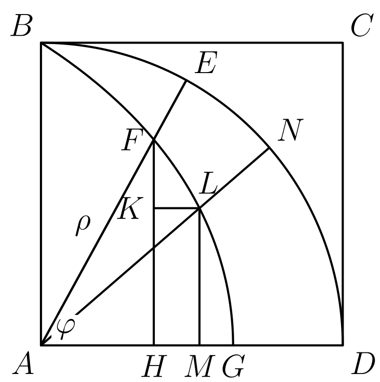
The plural *quadratrices* in the above passage is a Hellenism for the singlular *quadratrix*, which was a curve discovered by Hippias of Elis about 420 B.C. According to Proclus (p. 356, 11) Hippias proved its properties; and we are told (1) in the passage quoted above that Nicomedes also investigated it and that it was used for trisecting an angle, and (2) by Pappus (IV. pp. 250, 33–252, 4) that it was used by Dinostratus and Nicolmedes and some more recent writers for squaring the circle, whence its name. It is described thus (Pappus IV. p. 252.)

Suppose that $ABCD$ is a square and BED a quadrant of a circle with centre A.

Suppose (1) that a radius of the circle moves uniformly about A from the position AB to the position AD, and (2) that, *in the same time* the line BC moves uniformly, always parallel to itself, and with its extremity B moving along BA, from the position BC to the position AD.

Then the radius AE and the moving line BC determine at any instant by their intersection a point F.

The locus of F is the *quadratrix*.



The property of the curve is that, if F is any point, the arc BED is to the arc ED as AB is to FH.

In other words, if φ is the angle FAD, ρ the radius vector AF and a the side of the square,

$$(\rho \sin\varphi)/a = \varphi/\tfrac{1}{2}\pi.$$

Now the angle EAD can not only be *trisected* but *divided in any given ratio* by means of the quadratrix (Pappus IV. p. 286).

For let FH be divided at K in the given ratio.

Draw KL parallel to AD, meeting the curve in L; join AL and produce it to meet the circle in N.

Then the angles EAN, NAD are in the ratio of FK to KH, as is easily proved.

(c) Use of the *spiral of Archimedes*.

The trisection of an angle, or the division of an angle in any ratio, by means of the *spiral of Archimedes* is of course an equally simple matter. Suppose any angle included between the two radii vectores OA, OB of the spiral, and let it be required to cut the angle AOB in a given ratio. Since the radius vector increases proportionally with the angle described by the vector which generates the curve (reckoned from the original position of the vector coinciding with the initial line to the particular position assumed), we only have to take the radius vector OB (the greater of the two OA, OB), mark of OC along it equal to OA, cut CB in the given ratio (at D say), and then draw the circle with centre O and radius OD cutting the spiral in E. Then OE will divide the angle AOB in the required manner.