[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 259–261 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 7.]

1–6. In an English translation of the enunciation of this proposition it is absolutely necessary, in order to make it intelligible, to insert some words which are not in the Greek. The reason is partly that the Greek enunciation is itself very elliptical, and partly that some words used in it conveyed more meaning than the corresponding words in English do. Particularly is this the case with οὐ συσταθήσονται ἐπί "there shall not be constructed upon," since συνίστασθαι is the regular word for constructing a *triangle* in particular. Thus a Greek would easily understand συσταθήσονται ἐπί as meaning the construction of two lines *forming a triangle on* a given straight line as base; whereas to "construct two straight lines on a straight line" is not in English sufficiently definite unless we explain that they are drawn from the *ends* of the straight line to *meet* at a point. I have had the less hesitation in putting in the words "from the extremities" because they are actually used by Euclid in the somewhat similar enunciation of I. 21.

How impossible a literal translation into English is, if it is convey the meaning of the enunciation intelligibly, will be clear from the following attempt to render literally: "On the same straight line there shall not be constructed two other straight lines equal, each to each, to the same two straight lines (terminating) at different points on the same side, having the same extremities as the original straight lines" (ἐπὶ τῆς αὐτῆς εὐθείας δύο ταῖς αὐταῖς εὐθείαις ἄλλαι δύο εὐθεῖαι ἴσαι ἑκατέρα ἑκατέρᾳ οὐ συσταθήσονται πρὸς ἄλλῳ καὶ ἄλλῳ σημείῳ ἐπὶ τὰ αὐτὰ μέρη τὰ αὐτὰ πέρατα ἔχουσαι ταῖς ἐξ ἀρχῆς εὐθείαις).

The reason why Euclid allowed himself to use, in this enunciation, language apparently so obscure is no doubt that the phraseology was traditional and therefore, vague as it was, had a conventional meaning which the contemporary geometer well understood. This is proved, I think, by the occurrence in Aristotle (*Meteorologica* III. 5, 376 a 2 sqq.) of the very same, evidently technical expressions. Aristotle is there alluding to the theorem given by Eutocius from Apollonius' *Plane Loci* to the effect that, if H, K be two fixed points and M such a variable point that the ratio of MH to MK is a given ratio (not one of equality), the locus of M is a circle. (For an account of this theorem see note on VI. 3 below.) Now Aristotle says "The lines drawn up from H, K in this ratio cannot be constructed to two different points of the semicircle A" (αἱ οὖν ἀπὸ τῶν ΗΚ ἀναγόμεναι γραμμαὶ ἐν τούτῳ τῷ λόγῳ οὐ συσταθήσονται τοῦ ἐφ᾽ ᾧ Α ἡμικυκλίου πρὸς ἄλλο καὶ ἄλλο σημεῖον).

If a paraphrase is allowed instead of a translation adhering as closely as possible to the original, Simson's is the best that could be found, since the fact that the straight lines form *triangles* on the same base is really conveyed in the Greek. Simson's enunciation is, *Upon the same base, and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and likewise those which are terminated at the other extremity.* Th. Taylor (the translator of Proclus) attacks Simson's alteration as "indiscreet" and as detracting from the beauty and accuracy of Euclid's enunciation which are enlarged upon by Proclus in his commentary. Yet, when Taylor says, "Whatever difficulty learners may find in conceiving this proposition abstractedly is easily removed by its exposition in the figure," he really gives his case away. The fact is that Taylor, always enthusiastic over his author, was nettled by Simson's slighting remarks on Proclus' comments on the proposition. Simson had said, with reference to Proclus' explanation of the bearing of the second part of I. 5 on I. 7, that it was not "worth while to relate his trifles at full length," to which Taylor retorts "But Mr. Simson was no philosopher; and therefore the greatest part of these Commentaries must be considered by him as trifles, from the want of a philosophic genius to comprehend their meaning, and a taste superior to that of a *mere mathematician*, to discover their beauty and elegance."

20. It would be natural to insert here the step "but the angle ACD is greater than the angle BCD. [*C.N.* 5.]"

21. **much greater,** literally "greater by much" (πολλῷ μείζων). Simson and those who follow him translate: "*much more then* is the angle BDC *greater* than the angle BCD," but the Greek for this would have to be πολλῷ (or πολὺ) μᾶλλόν ἐστι ... μείζων. πολλῷ μᾶλλον, however, though used by Apollonius, is not, apparently, found in Euclid or Archimedes.

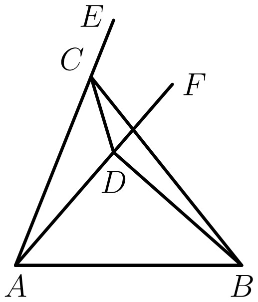
Just as in I. 6 we need a Postulate to justify theoretically the statement that CD falls within the angle ACB, so that the triangle DBC is less than the triangle ABC, so here we need Postulates which shall satisfy us as to the relative positions of CA, CB, CD on the one hand and of DC, DA, DB on the other, in order that we may be able to infer that the angle BDC is greater than the angle ADC, and the angle ACD greater than the angle BCD.

De Morgan (*op. cit.* p. 7) observes that I. 7 would be made easy to beginners if they were first familiarised, as a common notion, with "if any two magnitudes be equal, any magnitude greater than the one is greater than any magnitude less than the other." I doubt however whether a beginner would follow this easily; perhaps it would be more easily apprehended in the form "if any magnitude A is greater than a magnitude B, the magnitude A is greater than any magnitude equal to B, and (*a fortiori*) greater than any magnitude less than B."

It has been mentioned already (note on I. 5) that the second case of I. 7 given by Simson and in our text-books generally is not in the original text (the omission being in accordance with Euclid's general practice of giving only one case, and that the most difficult, and leaving the others to be worked out by the reader for himself). The second case is given by Proclus as the answer to a possible *objection* to Euclid's proposition, which should assert that the proposition is not proved to be universally true, since the proof given does not cover all possible cases. Here the objector is supposed to contend that what Euclid declares to be impossible may still be possible if one pair of lines lie wholly within the other pair of lines; and the second part of I. 5 enables the objection to be refuted.

If possible, let AD, DB be entirely within the triangle formed by AC, CB with AB, and let AC be equal to AD and BC to BD.

Join CD, and produce AC, AD to E and F.



Then, since AC is equal to AD,

the triangle ACD is isosceles,

and the angles ECD, FDC under the base are equal.

But the angle ECD is greater than the angle BCD,

therefore the angle FDC is also greater than the angle BCD.

Therefore the angle BDC is greater by far than the angle BCD.

Again, since DB is equal to CB,

the angles at the base of the triangle BDC are equal, [I. 5]

that is, the angle BDC is equal to the angle BCD.

Therefore the same angle BDC is both greater than and equal to the angle BCD: which is impossible.

The case in which D falls on AC or BC does not require proof.

I have already referred (note on I. 1) to the mistake made by those editors who regard I. 7 as being of no use except to prove I. 8. What I. 7 proves is that if, in addition to the base of a triangle, the length of the side terminating at each extremity of the base is given, only one triangle satisfying these conditions can be constructed on one and the same side of the given base. Hence not only does I. 7 enable us to prove I. 8, but it supplements I. 1 and I. 22 by showing that the constructions of those propositions give one triangle only on one and the same side of the base. But for I. 7 this could not be proved except by anticipating III. 10, of which therefore I. 7 is the equivalent for Book I. purposes. Dodgson (*Euclid and his modern Rivals*, pp. 194–5) puts it another way. "It [I. 7] shows that, of all plane figures that can be made by hingeing rods together, the *three*-sided ones (and these only) are *rigid* (which is another way of stating the fact that there cannot be *two* such figures on the same base). This is analogous to the fact, in relation to solids contained by plane surfaces hinged together, that *any* such solid is rigid, there being no maximum number of sides. And there is a close analogy between I. 7, 8 and III. 23, 24. these analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of." It will therefore be apparent how ill-advised are those editors who eliminate I. 7 altogether and rely on Philo's proof for I. 8.

Proclus, it may be added, gives (pp. 268, 19–269, 10) another explanation of the retention of I. 7, notwithstanding that it was apparently only required for I. 8. It was said that astronomers used it to prove that three successive eclipses could not occur at equal intervals of time, i.e. that the third could not follow the second at the same interval as the second followed the first; and it was argued that Euclid had an eye to this astronomical application of the proposition. But, as we have seen, there are other grounds for retaining the proposition which are quite sufficient of themselves.