[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 256–258 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 6.]

Euclid assumes that, because D is between A and B, the triangle DBC is less than the triangle ABC. Some postulate is necessary to justify this tacit assumption; considering an angle less than two right angles, say the angle ACB in the figure of the proposition, as a cluster of rays issuing from C and bounded by the rays CA, CB, and joining AB (where A, B are any two points on CA, CB respectively), we see that to each successive ray taken in the direction from CA to CB there corresponds one point on AB in which the said ray intersects AB, and that all the points on AB taken in order from A to B correspond univocally to all the rays taken in order from CA to CB, each point namely to the ray intersecting AB in the point.

We have here used, for the first time in the *Elements*, the method of *reductio ad absurdum*, as to which I would refer to the section above (pp. 136, 140) dealing with this among other technical terms.

This proposition also, being the *converse* of the preceding proposition, brings us to the subject of

Geometrical Conversion.

This must of course be distinguished from the *logical* conversion of a proposition. Thus, from the proposition that all isosceles triangles have the angles opposite to the equal sides equal, *logical* conversion would only enable us to conclude that *some* triangles with two angles equal are isosceles. Thus I. 6 is the geometrical, but not the logical, converse of I. 5. On the other hand, as De Morgan points out (*Companion to the Almanac*, 1849, p. 7), I. 6 is a purely *logical* deduction from I. 5 and I. 18 taken together, as is I. 19 also. For the general argument see the note on I. 19. For the present proposition it is enough to state the matter thus. Let X denote the class of triangles which have the two sides other than the base equal, Y the class of triangles which have the base angles equal; then we may call non-X the class of triangles having the sides other than the base unequal, non-Y the class of triangles having the base angles unequal.

Thus we have

All X is Y, [I. 5]
All non-X is non-Y; [I. 18]

and it is a purely logical deduction that

All Y is X. [I. 6]

According to Proclus (p. 252, 5 sqq.) two forms of *geometrical conversion* were distinguished.

(1) The leading form (προηγουμένη), the conversion *par excellence* (ἡ κυρίως ἀντιστροφή) is the complete or simple conversion in which the hypothesis and the conclusion of a theorem change places exactly, the conclusion of the theorem being the hypothesis of the converse thoerem, which again establishes, as its conclusion, the hypothesis of the original theorem. The relation between the first part of I. 5 and I. 6 is of this character. In the former the hypothesis is that two sides of a triangle are equal and the conclusion is that the angles at the base are equal, while the converse (I. 6) starts from the hypothesis that two angles are equal and proves that the sides subtending them are equal.

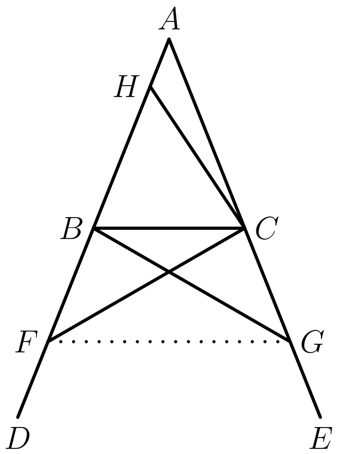
(2) The other form of conversion, which we may call *partial*, is seen in cases where a theorem states with two or more hypotheses combined into one enunciation and leads to a certain conclusion, after which the converse theorem takes this conclusion in substitution for one of the hypotheses of the original theorem and from the said conclusion along with the rest of the original hypotheses obtains, as its conclusion, the omitted hypothesis of the original theorem. I. 8 is in this sense a converse proposition to I. 4; for I. 4 takes as hypotheses (1) that two sides in two triangles are respectively equal, (2) that the included angles are equal, and proves (3) that the bases are equal, while I. 8 takes (1) and (3) as hypotheses and proves (2) as its conclusion. It is clear that a conversion of the *leading* type must be unique, while there may be many *partial* conversions of a theorem according to the number of hypotheses from which it starts.

Further, of convertible theorems, those which took as their hypothesis the *genus* and proved a *property* were distinguished as the *leading* theorems (προηγούμενα), while those which started from the property as hypothesis and described, as the conclusion, the genus possessing that property were the *converse* theorems. I. 5 is thus the leading theorem and I. 6 its converse, since the genus is in this case taken to be the isosceles triangle.

Converse of second part of I. 5.

Why, asks Proclus, did not Euclid convert the *second* part of I. 5 as well? He suggests, properly enough, two reasons: (1) that the second part of I. 5 itself is not wanted for any proof occurring in the original text, but is only put in to enable *objections* to the existing form of later propositions to be met, whereas the converse is not even wanted for this purpose; (2) that the converse could be deduced from I. 6, if wanted, at any time after we have passed I. 13, which can be used to prove that, if the angles formed by producing two sides of a triangle beyond the base are equal, the base angles themselves are equal.

Proclus adds a proof of the converse of the second part of I. 5, i.e., of the proposition that, if the angles formed by producing two sides of a triangle beyond the base are equal, the triangle is isosceles; but it runs to some length and then only effects a reduction to the theorem of I. 6 as we have it. As the result of this should hardly be assumed, a better proof would be an independent one *adapting* Euclid's own method in I. 6. Thus, with the construction of I. 5, we first prove by means of I. 4 that the triangles BFC, CGB are equal in all respects, and therefore that FC is equal to GB, and the angle BFC equal to the angle CGB. Then we have to prove that AF, AG are equal. If they are not, let AF be the greater, and from FA cut off FH equal to GA. Join CH.



Then we have, in the two triangles HFC, AGB,

two sides HF, FC equal to two sides AG, GB

and the angle HFC equal to the angle AGB.

Therefore (I. 4) the triangles HFC, AGB are equal. But the triangles BFC, CGB are also equal.

Therefore (if we take away these equals respectively) the triangles HBC, ACB are equal: which is impossible.

Therefore AF, AG are not unequal.

Hence AF is equal to AG and, if we subtract the equals BF, CG respectively, AB is equal to AC.

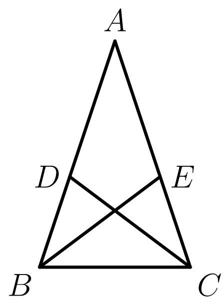
This proof is found in the commentary of an-Nairīzī (ed. Besthorn-Heiberg, p. 61, ed. Curtze, p. 50).

Alternative proofs of I. 6.

Todhunter points out that I. 6, not being wanted till II. 4, could be postponed till later and proved by means of I. 26. Bisect the angle BAC by a straight line meeting the base at D. Then the triangles ABD, ACD are equal in all respects.

Another method depending on I. 26 is given by an-Nairīzī after that proposition.

Measure equal lengths BD, CE along the sides BA, CA. Join BE, CD.



Then [I. 4] the triangles DBC, ECB are equal in all respects;
therefore EB, DC are equal, and the angles BEC, CDB are equal.

The supplements of the latter angles are equal [I. 13], and hence the triangles ABE, ACD have two angles equal respectively and the side BE equal to the side CD.

Therefore [I. 26] AB is equal to AC.