[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 250–255 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 5.]

2. the **equal straight lines** (meaning the equal *sides*). Cf. note on the similar expression in Prop. 4, lines 2, 3.

10. **Let a point F be taken at random on BD**, εἰλήφθω ἐπὶ τῆς ΒΔ τυχὸν σημεῖον τὸ Ζ, where τυχὸν σημεῖον means "a chance point."

17. **the two sides FA, AC are equal to the two sides GA, AB respectively**, δύο αἱ ΖΑ, ΑΓ δυσὶ ταῖς ΗΑ, ΑΒ ἴσαι εἰσὶν ἑκατέρα ἑκατατέρᾳ. Here, and in numberless later passages, I have inserted the word "sides" for the reason given in the note on I. 1, line 20. It would have been permissible to supply either "straight lines" or "sides"; but on the whole "sides" seems to be more in accordance with the phraseology of I. 4.

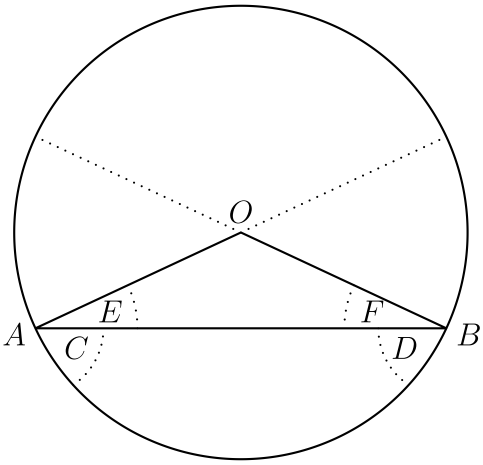
33. **the base BC is common to them**, i.e., apparently, common to the *angles*, as the αὐτῶν in βάσις αὐτῶν κοινή can only refer to γωνία and γωνίᾳ preceding. Simson wrote "and the base *BC* is common to the two triangles *BFC*, *CGB*"; Todhunter left out these words as being of no use and tending to perplex a beginner. But Euclid evidently chose to quote the conclusion of I. 4 exactly; the first phrase of that conclusion is that the bases (of the two triangles) are equal, and, as the equal bases are here the *same* base, Euclid naturally substitutes the word "common" for "equal."

48. As "(Being) what it was required to prove" (or "do") is somewhat long, I shall henceforth write the time-honoured "Q. E. D." and "Q. E. F." for ὅπερ ἔδει δεῖξαι and ὅπερ ἔδει ποιῆσαι.

According to Proclus (p. 250, 20) the discoverer of the fact that in any isosceles triangle the angles at the base are equal was Thales, who however is said to have spoken of the angles as being *similar*, and not as being *equal*. (Cf. Arist. *De caelo* IV. 311 b 34 πρὸς ὁμοίας γωνίας φαίνεται φερόμενον where *equal* angles are meant.)

A pre-Euclidean proof of I. 5.

One of the most interesting of the passages in Aristotle indicating differences between Euclid's proofs and those with which Aristotle was familiar, in other words, those of the text-books immediately preceding Euclid's has reference to the theorem of I. 5. The passage (*Anal. Prior.* I. 24, 41 b 13-22) is so important that I must quote it in full. Aristotle is illustrating the fact that in any syllogism one of the propositions must be affirmative and *universal* (καθόλου). "This," he says, "is better shown in the case of geometrical propositions" (ἐν τοῖς διαγράμμασιν), e.g. the proposition that *the angles at the base of an isosceles triangle are equal.*



"For let A, B be drawn [i.e. joined] to the centre.

"If, then we assumed (1) that the angle AC [i.e. $A + C$] is equal to the angle BD [i.e., $B + D$] without asserting generally that *the angles of semicircles are equal*, and again (2) that the angle C is equal to the angle D without making the further assumption that *the two angles of all segments are equal*, and if we then inferred, lastly, that, since the whole angles are equal, and equal angles are subtracted from them, the angles which remain, namely E, F, are equal, we should commit a *petitio principii*, unless we assumed [generally] that, *when equals are subtracted from equals, the remainders are equal*."

The language is noteworthy in some respects.

(1) A, B are said to be *drawn* (ἠγμένσι) to the centre (of the circle of which the two equal sides are radii) as if A, B were not the angular points but the sides or the radii themselves. (There is a parallel for this in Eucl. IV. 4.)

(2) "The angle AC" is the angle which is the sum of A and C, and A means here the angle at A of the *isosceles triangle* shown in the figure, and afterwards spoken of by Aristotle as E, while C is the "mixed" angle between AB and the circumference of the smaller segment cut off by it.

(3) The "angle *of* a semicircle" (i.e., the "angle" between the diameter and the circumference, at the extremity of the diameter) and the "angle *of* a segment" appear in Euclid III. 16 and III. Def. 7 respectively, obviously as survivals from earlier text-books.

But the most significant facts to be gathered from the extract are that in the text-books which preceded Euclid's "mixed" angles played a much more important part than they do in Euclid, and, in particular, that at least two propositions concerning such angles appeared quite at the beginning, namely the propositions that *the (mixed) angles of semicircles are equal*, and that *the two (mixed) angles of any segent of a circle are equal.* The wording of the first of the two propositions is vague, but it does not necessary mean more than that the two (mixed) angles in *one* semicircle are equal, and I know of no evidence going to show that it asserts that the angle of any one semicircle is equal to the angle of any other semicircle (of different size). It is quoted in the same form, "because the angles of semicircles are equal," in the Latin translation from the Arabic of Heron's *Catoptrica*, Prop. 9 (Heron, Vol. II., Teubner, p. 334), but it is only inferred that the different radii of *one* circle make equal "angles" with the circumference; and in the similar proposition of the Pseudo-Euclidean *Catoptrica* (Euclid, Vol. VII., p. 294) angles of the same sort in *one* circle are said to be equal "because they are (angles) of a semicircle." Therefore the first of the two propositions may be only a particular case of the second.

But it is remarkable enough that the second proposition (that *the two "angles of" any segment of a circle are equal*) should, in earlier text-books, have been placed before the theorem of Eucl. I. 5. We can hardly suppose it to have been proved otherwise than by the superposition of the semicircles into which the circle is divided by the diameter which bisects at right angles the base of the segment; and no doubt the proof would be closely connected with that of Thales' other proposition that any diameter of a circle bisects it, which must also (as Proclus indicates) have been proved by superposing one of the two parts upon the other.

It is a natural inference from the passage of Aristotle that Euclid's proof of I. 5 was his own, and it would thus appear that his innovations as regards order of propositions and methods of proof began at the very threshold of the subject.

Proof without producing the sides.

In this proof, given by Proclus (pp. 248, 22–249, 19), D and E are taken on AB, AC, instead of on AB, AC, *produced*, so that AD, AE are equal. The method of proof is of course exactly like Euclid's, but it does not establish the equality of the angles beyond the base as well.

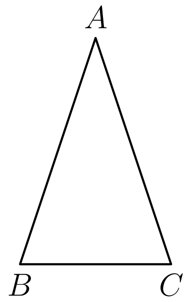
Pappus' proof

Proclus (pp. 249, 20–250,12) says that Pappus proved the theorem in a still shorter manner without the help of any construction whatever.

This very interesting proof is given as follows:

"Let ABC be an isosceles triangle, and AB equal to AC.

Let us conceive this one triangle as two triangles, and let us argue in this way.



Since AB is equal to AC, and AC to AB, the two sides AB, AC are equal to the two sides AC, AB.

And the angle BAC is equal to the angle CAB, for it is the same.

Therefore all the corresponding parts (in the triangles) are equal, namely

BC to BC,
the triangle ABC to the angle ABC (i.e. ACB),
the angle ABC to the angle ACB,
and the angle ACB to the angle ABC,

(for these are the angles subtended by the equal sides AB, AC).

Therefore in isosceles triangles the angles at the base are equal."

This will no doubt be recognised as the foundation of the alternative proof frequently given by modern editors, though they do not refer to Pappus. But they state the proof in a different form, the common method being to suppose the triangle to be taken up, turned over, and placed again upon *itself*, after which the same considerations of congruence as those used by Euclid in I. 4 are used over again. There is the obvious difficulty that it supposes the triangle to be taken up and at the same time to remain where it is. (Cf. Dodgson's humorous remark on this, *Euclid and his modern Rivals*, p. 47.) Whatever we may say in justification of the proceeding (e.g., that the triangle may be supposed to leave a *trace*), it is really equivalent to assuming the construction (hypothetical, if you will) of another triangle equal in all respects to the given triangle; and such an assumption is not in accordance with Euclid's principles and practice.

It seems to me that the form given to the proof by Pappus himself is by far the best, for the reasons (1) that it assumes no construction of a second triangle, real or hypothetical, (2) that it avoids the distinct awkwardness involved by a proof which, instead of merely quoting and applying the *result* of a previous proposition, repeats, with reference to a new set of data, the *process* by which that result was established. If it is asked how we are to realise Pappus' idea of *two* triangles, surely we may answer that we keep to one triangle and merely view it in two aspects. If it were a question of helping a beginner to understand this, we might say that one triangle is the triangle looked at in front and that the other triangle is the *same* triangle looked at from *behind*; but even this is not really necessary.

Pappus' proof, of course, does not include the proof of the second part of the proposition about the angles under the base, and we should still have to establish this much in the same way that Euclid does.

Purpose of the second part of the theorem.

An interesting question arises as to the reason for Euclid's insertion of the second part, to which, it will be observed, the converse proposition I. 6 has nothing corresponding. As a matter of fact, it is not necessary for any subsequent demonstration that is to be found in the original text of Euclid, but only for the interpolated second case of I. 7; and it was perhaps not unnatural that the undoubted genuineness of the second part of I. 5 convinced many editors that the second case of I. 7 must necessarily be Euclid's also. Proclus' explanation, which must apparently be the right one, is that the second part of I. 5 was inserted for the purpose of fore-arming the learner against a possible *objection* (ἔνστασις), as it was technically called, which might be raised to I. 7 as given in the text, with one case only. The *objection* would, as we have seen, take the specific ground that, as demonstrated, the theorem was not conclusive, since it did not cover all possible cases. From this point of view, the second part of I. 5 is useful not only for I. 7 but, according to Proclus, for I. 9 also. Simson does not seem to have grasped Proclus' meaning, for he says, "and Proclus acknowledges, that the second part of Prop. 5 was added upon account of Prop. 7 but gives a ridiculous reason for it, 'that it might afford an answer to objections made against the 7th,' as if the case of the 7th which is left out were, as he expressly makes it, an objection against the proposition itself."