[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 248–250 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Proposition 4.]

1–3. It is a fact that Euclid's enunciations not infrequently leave something to be desired in point of clearness and precision. Here he speaks of the triangles having "the angle equal to the angle, namely the angle contained by the equal straight lines" (τὴν γωνίαν τῇ γωνίᾳ ἴσαν ἔχῃ τὴν ὑπό τῶν ἴσων εὐθειῶν περιεχομένην), only one of the two angles being described in the latter expression (in the accusative), and a similar expression in the dative being left to be understood of the other angle. It is curious too that, after mentioning two "*sides*," he speaks of the angles contained by the equal "*straight lines*," not "*sides*." It may be that he wished to adhere scrupulously, at the outset, to the phraseology of the definitions, where the angle is the inclination to one another of the two *lines* or *straight lines*. Similarly in the enunciation of I. 5 he speaks of producing the equal "straight lines" as if to keep strictly to the wording of Postulate 2.

2. **respectively.** I agree with Mr H. M. Taylor (*Euclid*, p. ix) that it is best to abandon the traditional translation of "each to each," which would naturally seem to imply that all the four magnitudes are equal rather than (as the Greek ἑκατέρα ἑκατέρᾳ does) that one is equal to one and the other to the other.

3. **the base.** Here we have the word *base* used for the first time in the *Elements*. Proclus explains it (p.236, 12–15) as meaning (1), when no side of a triangle has been mentioned before, the side "which is on a level with the sight" (τὴν πρὸς τῇ ὄψει κειμένην), and (2), when two sides have already been mentioned, the third side. Proclus thus avoids the mistake made by some modern editors who explain the term exclusively with reference to the case where two sides have been mentioned before. That this is an error is proved (1) by the occurrence of the term in the enunciations of I. 37 etc. about triangles on the same base and equal bases, (2) by the application of the same term to the bases of parallelograms in I. 35 etc. The truth is that the use of the term must have been suggested by the practice of drawing the particular side horizontally, as it were, and the rest of the figure above it. The *base* of a figure was therefore spoken of, primarily, in the same sense as the base of anything else, e.g., of a pedestal or column; but when, as in I. 5, two triangles were compared occupying other than the normal positions which gave rise to the name, and when two sides had been previously mentioned, the base was, as Proclus says, necessarily the third side.

6. **subtend.** ὑποτείνειν ὑπό, "to stretch under," with accusative.

9. **the angle BAC.** The full Greek expression would be ἡ ὑπὸ τῶν ΒΑ, ΑΓ περιεχομένη γωνία, "the angle contained by the (straight lines) BA, AC." But it was common practice of Greek geometers, e.g. of Archimedes and Apollonius (and Euclid too in Books X.–XIII., to use the abbreviation αἱ ΒΑΓ for αἱ ΒΑ, ΑΓ, "the (straight lines) BA, AC." Thus, on περιεχομένη being dropped, the expression would become first ἡ ὑπὸ τῶν ΒΑΓ γωνία, then ἡ ὑπὸ ΒΑΓ γωνία, and finally ἡ ὑπὸ ΒΑΓ, without γωνία, as we regularly find it in Euclid.

17. **if the triangle be applied to...**, 23. **coincide.** The difference between the technical use of the passive ἐφαρμόζεσθαι "to be *applied* (to)," and of the active ἐφαρμόζειν "to *coincide* (with)" has been noticed above (note on *Common Notion* 4, pp. 224–5).

32. [**For if, when B coincides...** 36. **coincide with EF**]. Heiberg (*Paralipomena zu Euklid* in *Hermes*, XXXVIII., 1903, p. 56) has pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of an-Nairīzī (*Codex Leidensis* 399, I, ed. Besthorn-Heiberg, p. 55) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a *scholium* only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since B coincides with E and C with F, the straight line BC coincides with the straight line EF, an inference which readily follows from the definition of a straight line and Post. 1; and no doubt the Postulate that "Two straight lines cannot enclose a space" (afterwards placed among the *Common Notions*) was interpolated at the same time.

44. **Therefore etc.** Where (as here) Euclid's *conclusion* merely repeats the enunciation word for word, I shall avoid the repetition and write "Therefore etc." simply.

In the note on *Common Notion* 4 I have already mentioned that Euclid obviously used the method of superposition with reluctance, and I have given, after Veronese for the most part, the reason for holding that that method is not admissible as a *theoretical* means of proving equality, although it may be of use as a *practical* test, and may thus furnish an empirical basis on which to found a postulate. Mr. Bertrand Russell observes (*Principles of Mathematics* I. p. 405) that Euclid would have done better to assume I. 4 as an axiom, as is practically done by Hilbert (*Grundlagen der Geometrie*, p. 9). It may be that Euclid himself was as well aware of the objections to the method as are his modern critics; but at all events those objections were stated, with almost equal clearness, as early as the middle of the 16th century. Peletarius (Jacques Peletier) has a long note on this proposition (*In Euclidis Elementa geometrica demonstrationum libri sex*, 1557), in which he observes that, if superposition of lines and figures could be assumed as a method of proof, the whole of geometry would be full of such proofs, that it could equally well have been used in I. 2, 3 (thus in I. 2 we could simply have supposed the line taken up and *placed* at the point), and that in short it is obvious how far removed the method is from the dignity of geometry. The theorem, he adds, is obvious in itself and does not require proof; although it is introduced as a theorem, it would seem that Euclid intended it rather as a *definition* than a theorem, "for I cannot think that two angles are equal unless I have a conception of what equality of angles is." Why then did Euclid include the proposition among theorems, instead of placing it among the axioms? Peletarius makes the best excuse he can, but concludes thus:

"Huius itaque propositionis veritatem non aliunde quam a communi iudicio petemus; cogitabimusque figuras figuris superponere, Mechanicum quippiam esse: intelligere verò, id demum esse Mathematicum."

Expressed in terms of the modern systems of Congruence-Axioms referred to in the note on *Common Notion* 4, what Euclid really assumes amounts to the following:

(1) On the line DE, there is a point E, on either side of D, such that AB is equal to DE.

(2) On either side of the ray DE there is a ray DF such that the angle EDF is equal to the angle BAC.

It now follows that on DF there is a point F such that DF is equal to AC.

And lastly (3), we require an axiom from which to infer that the two remaining angles of the triangles are respectively equal and that the bases are equal.

I have shown above (pp. 229–230) that Hilbert has an axiom stating the equality of the remaining angles simply, but proves the equality of the bases.

Another alternative is that of Pasch (*Vorlesungen über neuere Geometrie*, p.109) who has the following "Grundsatz":

If two figures AB and FGH are given (FGH not being contained in a straight length), and AB, FG are congruent, and if a plane surface be laid through A and B, we can specify in this plane surface, produced if necessary, two points C, D, neither more nor less, such that the figures ABC and ABD are congruent with the figure FGH, and the straight line CD has with the straight line AB or with AB produced one point common.

I pass to two points of detail in Euclid's proof:

(1) The inference that, since B coincides with E, and C with F, the bases of the triangles are wholly coincident rests, as expressly stated, on the impossibility of two straight lines enclosing a space, and therefore presents no difficulty.

But (2) most editors seem to have failed to observe that at the very beginning of the proof a much more serious assumption is made without any explanation whatever, namely that, if A be placed on D, and AB on DE, the point B will coincide with E, because AB is equal to DE. That is, the *converse* of *Common Notion* 4 is assumed for straight lines. Proclus merely observes, with regard to the converse of this Common Notion, that it is only true in the case of things "of the same form" (ὁμοειδῆ), which he explains as meaning straight lines, arcs of one and the same circle, and angles "contained by lines similar and similarly situated" (p. 241, 3–8).

Savile however saw the difficulty and grappled with it in his note on the Common Notion. After stating that all straight lines with two points common are congruent between them (for otherwise two straight lines would enclose a space), he argues thus. Let there be two straight lines AB, DE, and let A be placed on D, and AB on DE. Then B will coincide with E. For, if not, let B fall somewhere short of E or beyond E; and in either case it will follow that the less is equal to the greater, which is impossible.

Savile seems to assume (and so apparently does Lardner who gives the same proof) that, if the straight lines be "applied," B will fall somewhere on DE or DE produced. But the grounds for this assumption should surely be stated; and it seems to me that it is necessary to use, not Postulate 1 alone, nor Postulate 2 alone, but both, for this purpose (in other words to assume, not only that *two straight lines cannot enclose a space*, but also that *two straight lines cannot have a common segment*). For the only safe course is to place A upon D and then turn AB about D until *some* point on AB intermediate between A and B coincides with *some* point on DE. In this position AB and DE have two points common. Then Postulate 1 enables us to infer that the straight lines coincide *between* the two common points, and Postulate 2 that they coincide beyond the second common point towards B and E. Thus the straight lines coincide throughout so far as *both* extend; and Savile's argument then proves that B coincides with E.