[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 202–220 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Postulate 5.]

Postulate 5.

Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο ευθείας ἐπ᾽ ἄπειρον συμπίπτειν ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Although Aristotle gives a clear idea of what he understood by a *postulate*, he does not give any instances from geometry; still less has he any allusion recalling the particular postulates found in Euclid. We naturally infer that the formulation of these postulates was Euclid's own work. There is a more positive indication of the originality of Postulate 5, since in the passage (*Anal. prior.* II. 16, 65 a 4) quoted above in the note on the definition of parallels he alludes to some *petito principii* involved in the theory of parallels current in his time. This reproach was removed by Euclid when he laid down this epoch-making Postulate. When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot but admire the genius of the man who concluded that such a hypothesis, which he found necessary to the validiy of his whole system of geometry, was really indemonstrable.

From the very beginning, as we know from Proclus, the Postulate was attacked as such, and attempts were made to prove it as a theorem or to get rid of it by adopting some other definition of parallels; while in modern times the literature of the subject is enormous. Riccardi (*Saggio di una bibliografia Euclidea*, Part IV., Bologna, 1890) has twenty quarto pages of titles of monographs relating to Post 5 between the dates 1607 and 1887. Max Simon (*Ueber di Entwicklung der Elementar-geometrie im XIX. Jahrhundert*, 1906) notes that he has seen three new attempts as late as 1891 (a century after Gauss laid the foundation of non-Euclidean geometry), to prove the theory of parallels independently of the Postulate. Max Simon himself (pp. 53–61) gives a large number of references to books or articles on the subject and refers to the copious information, as to contents as well as names, contained in Schotten's *Inhalt und Methode des planimetrischen Unterrichts*, II. pp. 183–332.

This note will include some account of or allusion to a few of the most noteworthy attempts to prove the Postulate. Only those of ancient times, as being less generally accessible, will be described at any length; shorter references must suffice in the case of the modern geometers who have made the most important contributions to the discussion of the Postulate and have thereby, in particular, contributed most towards the foundation of the non-Euclidean geometries, and here I shall make use principally of the valuable Article 8, *Sulla teoria delle parallele e sulle geometrie non-euclidee* (by Roberto Bonola), in *Questioni riguardanti le matematiche elementari*, I. pp. 247–363.

Proclus (p. 191, 21 sqq.) states very clearly the nature of the first objections taken to the Postulate.

"This ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties, which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid himself as a theorem. It may be that some would be deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for an instantaneous belief that the straight lines converge and meet. To such as these Geminus correctly replied that we have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. For Aristotle says that it is as justifiable to ask scientific proofs of a rhetorician as to accept mere plausibilities from a geometer; and Simmias is made by Plato to say that he recognises as quacks those who fashion for themselves proofs from probabilities. So in this case the fact that, when the right angles are lessened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometimes meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely, but yet remain non-secant (ἀσύμπτωτοι), although it seems improbable and paradoxical, is nevertheless true and fully ascertained with regard to other species of lines. May not then the same thing be possible in the case of straight lines which happens in the case of the lines referred to? Indeed, until the statement in the Postulate is clinched by proof, the facts shown in the case of other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unreasoned (hypothesis)?

"It is then clear from this that we must seek a proof of the present theorem, and that it is alien to the special character of postulates. But how it should be proved, and by what sort of arguments the objections taken to it should be removed, we must explain at the point where the writer of the Elements is actually about to recall it and use it as obvious. It will be necessary at that stage to show that its obvious character does not appear independently of proof, but is turned by proof into matter of knowledge."

Before passing to the attempts of Ptolemy and Proclus to prove the Postulate, I should note here that Simplicius says (in an-Nairīzī, ed. Besthorn-Heiberg, p. 119, ed. Curtze, p. 65) that this Postulate is by no means manifest, but requires proof, and accordingly "Abthiniathus" and Diodorus had already proved it by means of many different propositions, while Ptolemy also had explained and proved it, using for the purpose Eucl. I. 13, 15 and 16 (or 18). The Diodorus here mentioned may be the author of the *Analemma* on which Pappus wrote a commentary. It is difficult even to frame a conjecture as to who "Abthiniathus" is. In one place in the Arabic text the name appears to be written "Anthisathus" (H. Suter in *Zeitschrift für Math. und Physik*, XXXVIII., hist. litt. Abth. p. 194). It has occurred to me whether he might be Peithon, a friend of Serenus of Antinoeia (Antinoupolis) who was long known as Serenus of *Antissa*. Serenus says (*De sectione cylindri*, ed. Heiberg, p. 96): "Peithon the geometer, explaining parallels in a work of his, was not satisfied with what Euclid said, but showed their nature more cleverly by an example; for he says that parallel straight lines are such a thing as we see on walls or on the ground in the shadows of pillars which are made when either a torch or a lamp is burning behind them. And, although this has only been matter of merriment to every one, I at least must not deride it, for the respect I have for the author, who is my friend." If Peithon was known as "of Antinoeia" or "of Antissa," the two forms of the mysterious name might perhaps be an attempt at an equivalent; but this is no more than a guess.

Simplicius adds in full and word for word the attempt of his "friend" or his "master Aganis" to prove the Postulate.

Proclus returns to the subject (p. 365, 5) in his note on Eucl. I. 29. He says that before his time a certain numbr of geometers had classed as a theorem this Euclidean postulate and thought it matter for proof, and he then proceeds to give an account of Ptolemy's argument.

Noteworthy attempts to prove the Postulate.

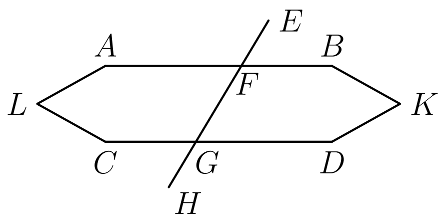
Ptolemy.

We learn from Proclus (p. 365, 7–11) that Ptolemy wrote a book on the proposition that "straight lines drawn from angles less than two right angles meet if produced," and that he used in his "proof" many of the theorems in Euclid preceding I. 29. Proclus excuses himself from reproducing the early part of Ptolemy's argument, only mentioning as one of the propositions proved in it the theorem of Eucl. I. 28 that, if two straight lines meeting a transversal make the two interior angles on the same side equal to two right angles, the straight lines do not meet, however far produced.

I. From Proclus' note on I. 28 (p. 362, 14 sq.) we know that Ptolemy proved this somewhat as follows.

Suppose that there are two straight lines AB, CD, and that $EFGH$, meeting them, makes the angles BFG, FGD equal to two right angles. I say that AB, CD are parallel, that is, they are non-secant.

For, if possible, let FB, GD meet at K.



Now, since the angles BFG, FGD are equal to two right angles, while the four angles AFG, BFG, FGD, FGC are together equal to four right angles, the angles AFG, FGC are equal to two right angles.

"*If therefore* FB, GD, *when the interior angles are equal to two right angles, meet at* K, *the straight lines* FA, GC *will also meet if produced;* for the angles AFG, CGF are also equal to two right angles.

"Therefore, the straight lines will either meet in both directions or in neither direction, if the two pairs of interior angles are both equal to two right angles.

"Let, then, FA, GC meet at L.

"Therefore the straight lines $LABK$ $LCDK$ enclose a space: which is impossible.

"Therefore it is not possible for two straight lines to meet when the interior angles are equal to two right angles. Therefore they are parallel."

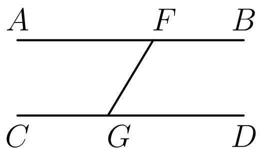
[The argument in the words italicised would be clearer if it had been shown that the two interior angles on one side of EH are *severally* equal to the two interior angles on the other, BFG to CGF and FGD to AFG; when, assuming FB, GD to meet in K, we can take the triangle KFG and place it (e.g. by rotating it in the plane about O the middle point of FG) so that FG falls where GF is in the figure and GD falls on FA, in which case FB must also fall on GC; hence, since FB, GD meet at K, GC and FA must meet at a corresponding point L. Or, as Mr Frankland does, we may substitute for FG a straight line MN through O the middle point of FG drawn perpendicular to one of the parallels, say AB. Then, since the two triangles OMF, ONG have two angles equal respectively, namely FOM to GON (I. 15) and OFM to OGN, and one side OF equal to one side OG, the triangles are congruent, the angle ONG is a right angle, and MN is perpendicular to both AB and CD. Then, by the same method of application, MA, NC are shown to form with MN a triangle $MALCN$ congruent with the triangle $NDKBM$, and MA, NC meet at a point L corresponding to K. Thus the two straight lines would meet at the *two* points K, L. This is what happens under the Riemann hypothesis, where the axiom that two stright lines cannot enclose a space does not hold, but all straight lines meeting in one point have another point common also, and e.g. in the particular figure just used K, L are points common to all perpendiculars to MN. If we suppose that K, L are not distinct points, but *one* point, the axiom that two straight lines cannot enclose a space is *not* contradicted.]

II. Ptolemy now tries to prove I. 29 without using our Postulate, and then deduces the Postulate from it (Proclus, pp. 365, 14—367, 27).

The argument to prove I. 29 is as follows.

The straight line which cuts the parallels must make the sum of the interior angles on the same side equal to greater than, or less than, two right angles.

"Let AB, CD be parallel, and let FG meet them. I say (1) that FG does not make the interior angles on the same side greater than two right angles.



"For, if the angles AFG, CGF are greater than two right angles, the remaining angles BFG, DGF are less than two right angles.

"But the same two angles are also greater than two right angles; *for AF, CG are no more parallel than FB, GD, so that, if the straight line falling on AF, CG makes the interior angles greater than two right angles, the straight line falling on FB, GD will also make the interior angles greater than two right angles.*

"But the same angles are also less than two right angles; for the four angles AFG, CGF, BFG, DGF are equal to four right angles: which is impossible.

“Similarly (2) we can show that the straight line falling on the parallels does not make the interior angles on the same side less than two right angles.

“But (3), if it makes them neither greater nor less than two right angles, it can only make the interior angles on the same side *equal* to two right angles.”

III. Ptolemy deduces Post. 5 thus:

Suppose that the straight lines making angles with a transversal less than two right angles do not meet on the side on which those angles are.

Then, *a fortiori*, they will not meet on the other side on which are the angles *greater* than two right angles.

Hence the straight lines will not meet in either direction; they are therefore *parallel*.

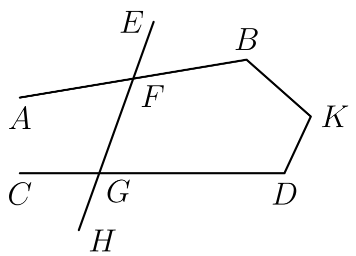
But, if so, the angles made by them with the transversal are equal to two right angles, by the preceding proposition (= I. 29).

Therefore the same angles will be both equal to and less than two right angles:
which is impossible.

Hence the straight lines will meet.

IV. Ptolemy lastly enforces his conclusion that the straight lines will meet *on the side on which are the angles less than two right angles* by recurring to the *a fortiori* step in the foregoing proof.

Let the angles AFG, CGF in the accompanying figure be together less than two right angles.



Therefore the angles BFG, DGF are greater than two right angles.

We have proved that the straight lines are not non-secant.

If they meet, they must meet either towards A, C, or towards B, D.

(1) Suppose they meet towards B, D, at K.

Then, since the angles AFG, CGF are less than two right angles, and the angles AFG, GFB are equal to two right angles, take away the common angle AFG, and the angle CGF is less than the angle BFG; that is, the exterior angle of the triangle KFG is less than the interior and opposite angle BFG: which is impossible.

Therefore AB, CD do not meet before B, D.

(2) But they do meet, and therefore they must meet in one direction or the other: therefore they meet towards A, B, that is, on the side where are the angles less than two right angles.

The flaw in Ptolemy's argument is of course in the part of his proof of I. 29 which I have italicised. As Proclus says, he is not entitled to assume that, if AB, CD are parallel, whatever is true of the interior angles on one side of FG (i.e. that they are together equal to, greater than, or less than, two right angles) is necessarily true at the same time of the interior angles on the other side. Ptolemy justifies this by saying that FA, GC are no more parallel in one direction than FB, GD are in the other: which is equivalent to the assumption that *through any point only one parallel can be drawn to a given straight line*. That is, he assumed an equivalent of the very Postulate he is endeavouring to prove.

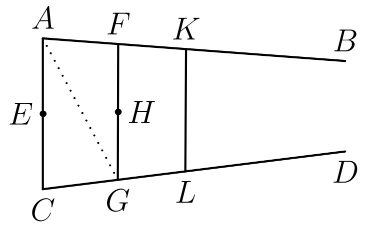
Proclus.

Before passing to his own attempt at a proof, Proclus (p. 368, 26 sqq.) examines an ingenious argument (recalling somewhat the famous one about Achilles and the tortoise) which appeared to show that it was *impossible* for the lines described in the Postulate to meet.

Let AB, CD make with AC the angles BAC, ACD together less than two right angles.

Bisect AC at E and along AB, CD respectively measure AF, CG so that each is equal to AE.

Bisect FG at H and mark off FK, GL each equal to FH; and so on.



Then AF, CG will not meet at any point on FG; for, if that were the case, two sides of a triangle would be together equal to the third: which is impossible.

Similarly, AB, CD will not meet at any point on KL; and "proceeding like this indefinitely, joining the non-coincident points, bisecting the lines so drawn, and cutting off from the straight lines portions equal to the half of these, they say they thereby prove that the straight lines AB, CD will not meet anywhere."

It is not surprising that Proclus does not succeed in exposing the fallacy here (the fact being that the process will indeed be endless, and yet the straight lines will intersect within a finite distance). But Proclus' criticism contains nevertheless something of value. He says that the argument will prove too much, since we have only to join AG in order to see that straight lines making *some* angles which are together less than two right angles do in fact meet, namely AG, CG. "Therefore it is not possible to assert, without some definite limitation, that the straight lines produced from angles less than two right angles do not meet. On the contrary, it is manifest that *some* straight lines, when produced from angles less than two right angles, do meet, although the argument seems to require it to be proved that this property belongs to *all* such straight lines. For one might say that, the lessening of the two right angles being subject to no limitatation, *with such and such an amount of lessening the striaght lines remain non-secant, but with an amount of lessening in excess of this they meet* (p. 371, 2–10)."

[Here then we have the germ of such an idea as that worked out by Lobachewsky, namely that the straight lines issuing from a point in the plane can be divided with reference to a straight line lying in that plane into two classes, "secant" and "non-secant," and that we may define as *parallel* the two straight lines which divide the secant from the non-secant class.]

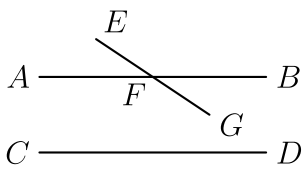
Proclus goes on (p. 371, 10) to base his own argument upon "an axiom such as Aristotle too used in arguing that the universe is finite. For, *if from one point two straight lines forming an angle be produced indefinitely will exceed any finite magnitude.* Aristotle at all events showed that, if the straight lines drawn from the centre to the circumference are infinite, the interval between them is infinite. For, if it is finite, it is impossible to increase the distance, so that the straight lines (the radii) are not infinite. Hence the straight lines, which when produced indefinitely, will be at a distance from one another greater than any assumed finite magnitude."

This is a fair representation of Aristotle's argument in *De caelo* I. 5, 271 b 28, although of course it is not a proof of what Proclus assumes as an axiom.

This being premised, Proclus proceeds (p. 371, 24):

I. "I say that, *if any straight line cuts one of two parallels, it will cut the other also.*

"For let AB, CD be parallel, and let EFG cut AB; I say that it will cut CD also.



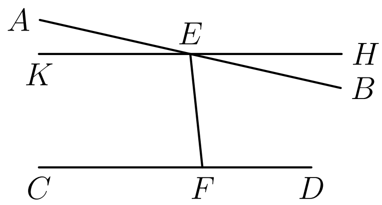
"For, since BF, FG are two straight lines from one point F, they have, when produced indefinitely, a distance greater than any magnitude, so that it will also be greater than the interval between the parallels. Whenever therefore they are at a distance from one another greater than the distance between the parallels, FG will cut CD.

"Therefore etc."

II. "Having prove this, we shall prove, as a deduction from it, the theorem in question.

"For let AB, CD be two straight lines, and let EF falling on them make the angles BEF, DFE less than two right angles.

"I say that the straight lines will meet on that side on which are the angles less than two right angles.



"For, since the angles BEF, DFE are less than two right angles, let the angle HEB be equal to the excess of two right angles (over them), and let HE be produced to K.

"Since then EF falls on KH, CD and makes the two interior angles HEF, DFE equal to two right angles, the straight lines HK, CD are parallel.

"And AB cuts KH; therefore it will also cut CD, but what was before shown.

"Therefore AB, CD will meet on that side on which are the angles less than two right angles.

"Hence the theorem is proved."

Clavius criticised this proof on the ground that the axiom from which it starts, taken from Aristotle, itself requires proof. He points out that, just as you cannot assume that two lines which continually approach one another will meet (witness the hyperbola and its asymptote), so you cannot assume that two lines which continually diverge will ultimately be so far apart that a perpendicular from a point on one let fall on the other will be greater than any assigned distance; and he refers to the *conchoid* of Nicomedes, which continally approaches its asymptote, and therefore continually gets farther away from the tangent at the vertex; yet the perpendicular from any point on the curve to that tangent will always be less than the distance between the tangent and the asymptote. Saccheri supports the objection.

Proclus' first proposition is open to the objection that it assumes that two "parallels" (in the Euclidean sense) or, as we may say, two straight lines which have a common perpendicular, are (not necessarily equidistant, but) so related that, when they are produced indefinitely, the perpendicular from a point of one upon the other remains finite.

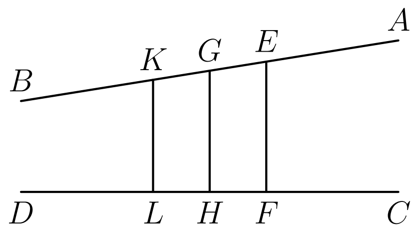
This last assumption is incorrect on the hyperbolic hypothesis; the "axiom" taken from Aristotle does not hold on the elliptic hypothesis.

Naṣīraddīn aṭ-Ṭūsī

The Persian-born editor of Euclid, whose date is 1201–1274, has three lemmas leading up to the final proposition. Their content is substantially as follows, the first lemma being apparently assumed as evident.

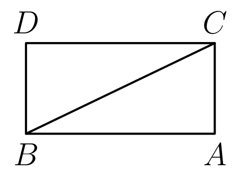
I. (a) If AB, CD be two straight lines such that successive perpendiculars, as EF, GH, KL, from points AB to CD always make with AB unequal angles, which are always acute on the side towards B and always obtuse on the side towards A, then the lines AB, CD, so long as they do not cut, approach continually nearer in the direction of the obtuse angles, and the perpendiculars diminish towards B, D, and increase towards A, C.

(b) Conversely, if the perpendiculars so drawn continually become shorter in the direction of B, D, and longer in the direction of A, C, the straight lines AB, CD approach continually nearer in the direction of B, D and diverge continually in the other direction; also each perpendicular will make with AB two angles one of which is acute and the other is obtuse, and all the acute angles will lie in the direction towards B, D, and the obtuse angles in the opposite direction.



[Saccheri points out that even the first part (a) requires proof. As regards the converse (b) he asks, why would not the successive acute angles made by the perpendiculars with AB, while remaining acute, become greater and greater as the perpendiculars become smaller until we arrive at last at a perpendicular which is a common perpendicular to *both* lines? If that happens, all the author's efforts are in vain. And, if you are to assume the truth of the statement in the lemma without proof, would it not, as Wallis said, be as easy to assume as axiomatic the statement in Post. 5 without more ado?]

II. *If* AC, BD *be drawn from the extremities of* AB *at right angles to it and on the same side, and if* AC, BD *be made equal to one another and* CD *be joined, each of the angles* ACD*,* BDC *will be right, and* CD *will be equal to* AB*.*



The first part of this lemma is proved by *reductio ad absurdum* from the preceding lemma. If, e.g., the angle ACD is not right, it must either be acute or obtuse.

Suppose it is acute; then, by lemma I, AC is greater than BD, which is contrary to the hypothesis. And so on.

The angles ACD, BDC being proved to be right angles, it is easy to prove that AB, CD are equal.

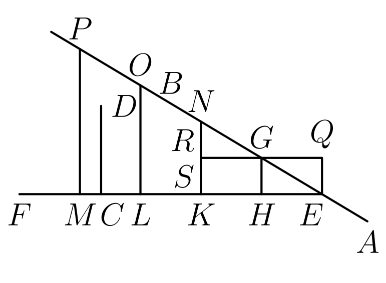
[It is of course assumed in this "proof" that, if the angle ACD is acute, the angle BDC is obtuse, and *vice versa*.]

III. *In any triangle the three angles are together equal to two right angles.* This is proved for a *right-angled* triangle by means of the foregoing lemma, the four angles of the quadrilateral $ABCD$ of that lemma being all right angles. The proposition is then true for *any* triangle, since any triangle can be divided into two right-angled triangles.

IV. Here we have the final "proof" of Post. 5. Three cases are distinguished, but it is enough to show the case where one of the interior angles is right and the other acute.

Suppose AB, CD to be two straight lines met by FCE making the angle ECD a right angle and the angle CEB an acute angle.

Take any point G on EB, and draw GH perpendicular to EC.



Since the angle CEG is acute, the perpendicular GH will fall on the side of E towards D, and will either coincide with CD or not coincide with it. In the former case the proposition is proved.

If GH does not coincide with CD but falls on the side of it towards F, CD, being within the triangle formed by the perpendicular and by CE, EG, must cut EG. [An axiom is here used, namely that, if CD be produced far enough, it must pass *outside* the triangle and therefore cut *some* side, which must be EB, since it cannot be the perpendicular (I. 27), or CE.]

Lastly, let GH fall on the side of CD towards E.

Along HC set off HK, KL etc., each equal to EH, until we get the first point of division, as M, beyond C.

Along GB set off GN, NO etc., each equal to EG, until EP is the same multiple of EG that EM is of EH.

Then we can prove that the perpendiculars from N, O, P on EC fall on the points K, L, M respectively.

For take the first perpendicular, that from N, and call it NS.

Draw EQ at right angles to EH and equal to GH, and set off SR along SN also equal to GH. Join QG, GR.

Then (second lemma) the angles EQG, QGH are right, and $QG = EH$.

Similarly the angles SRG, RGH are right, and $RG = SH$.

Thus RGQ is one straight line, and the vertically opposite angles NGR, EGQ are equal. The angles NRG, EQG are both right, and $NG = GE$, by construction.

Therefore (I. 26) $RG = GQ$; whence $SH = HE = KH$, and S coincides with K.

We may proceed similarly with the other perpendiculars.

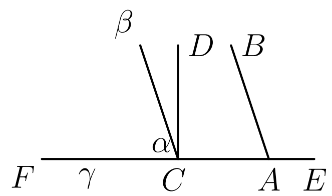
Thus PM is perpendicular to FE. Hence CD, being parallel to MP and within the triangle PME, must cut EP, if produced far enough.

John Wallis.

As is well known, the argument of Wallis (1616–1703) assumed as a postulate that, *given a figure, another figure is possible which is similar to the given one and of any size whatever.* In fact Wallis assumed this for *triangles* only. He first proved (1) that, if a finite straight line is placed on an infinite straight line, and is then moved in its own direction as far as we please, it will always line on the same infinite straight line, (2) that, if an angle be moved so that one leg always slides along an infinite stright line, the angle will remain the same, or equal, (3) that, if two straight lines, cut by a third, make the interior angles on the same side less than two right angles, each of the exterior angles is greater than the opposite interior angle (proved by means of I. 13).

(4) If AB, CD make, with AC, the interior angles less than two right angles, suppose AC (with AB rigidly attached to it) to move along AF to

the position $\alpha\gamma$, such that α coincides with C. If AB then takes the position $\alpha\beta$, $\alpha\beta$ lies entirely outside CD (proved by means of (3) above).



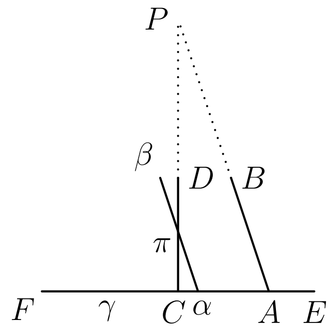
(5) With the same hypotheses, *the straight line* $\alpha\beta$*, or* AB*, during its motion, and before* α *reaches* C *must* cut *the straight line* CD*.*

(6) Here is enunciated the postulate stated above.

(7) Postulate 5 is now proved thus.

Let AB, CD be the straight lines which make, with the infinite straight line ACF meeting them, the interior angles BAC, DCA together less than two right angles.

Suppose AC (with AB rigidly attached to it) to move along ACF until AB takes the position of $\alpha\beta$ cutting CD in π.



Then $\alpha C\pi$ being a triangle, we can, by the above postulate, suppose a triangle drawn on the base CA similar to the triangle $\alpha C\pi$.

Let it be ACP.

[Wallis here interposes a defence of the hypothetical construction.]

Thus CP and AP meet at P; and as by the definition of similar figures the angles of the trinagles PCA, $\pi C\alpha$ are respectively equal, the angle PCA being equal to the angle $\pi C\alpha$ and the angle PAC to the angles $\pi\alpha C$ or BAC, it follows that CP, AP lie on CD, AB produced respectively.

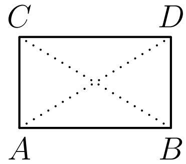
Hence AB, CD meet on the side on which are the angles less than two right angles.

[The whole gist of this proof lies in the assumed postulate as to the existence of similar figures; and, as Saccheri points out, this is equivalent to unconditionally assuming the "hypothesis of the right angle," and consequently Euclid's Postulate 5.]

Gerolamo Saccheri.

The book *Euclides ab omni naevo vindicatus* (1733) by Gerolamo Saccheri (1667–1733), a Jesuit, and professor at the University of Pavia, is now accessible (1) edited in German by Engle and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, pp. 41–136, and (2) in an Italian version, abridged but annotated, *L'Euclide emendato del P. Gerolamo Saccheri*, by G. Boccardini (Hoepli, Milan, 1904). It is of much greater importance than all the earlier attempts to prove Post. 5 because Saccheri was the first to contemplate the possibility of hypotheses other than that of Euclid, and to work out a number of consequences of those hypotheses. He was therefore a true precursor of Legendre and of Lobachewsky, as Beltrami called him (1889), and, it might be added, of Riemann also. For, as Veronese observes (*Fondamenti di geometria*, p. 570), Saccheri obtained a glimpse of the theory of parallels in all its generality, while Legendre, Lobachewsky and G. Bolyai, excluded *a priori*, without knowing it, the "hypothesis of the obtuse angle," or the Riemann hypothesis. Saccheri, however, was the victim of the preconceived notion of his time that the sole possible geometry was the Euclidean, and he presents the curious spectacle of a man laboriously erecting a structure upon new foundations for the very purpose of demolishing it afterwards; he sought for contradictions in the heart of the systems which he constructed, in order to prove thereby the falsity of his hypotheses.

For the purpose of formulating his hypotheses he takes a plane quadrilateral $ABDC$, two opposite sides of which AC, BD, are equal and perpendicular to a third AB. Then the angles at C and D are easily proved to be equal. On the Euclidean hypothesis they are both right angles; but apart from this hypothesis they might be both obtuse or both acute. To the three possibilities, which Saccheri distinguishes by the names (1) *the hypothesis of the right angle*, (2) *the hypothesis of the obtuse angle* and (3) *the hypothesis of the acute angle* respectively, there corresponds a certain group of theorems; and Saccheri's point of view is the Postulate will be completely proved if the consequences which follow from the last two hypotheses comprise results inconsistent with one another.



Among the most important of his propositions are the following:

(1) *If the hypothesis of the right angle, or of the obtuse angle, or of the acute angle is proved true in a single case, it is true in every other case.* (Props. V., VI., VII.)

(2) *According as the hypothesis of the right angle, the obtuse angle, or the acute angle is true, the sum of the three angles of a triangle is equal to, greater than, or less than two right angles.* (Prop. IX.)

(3) *From the existence of a single triangle in which the sum of the angles is equal to, greater than, or less than two right angles the truth of the hypothesis of the right angle, obtuse angle, or acute angle respectively follows.* (Prop. XV.)

These propositions involve the following: *If in a single triangle the sum of the angles is equal to, greater than, or less than two right angles, then* any *triangle has the sum of its angles equal to, greater than, or less than two right angles respectively*, which was proved about a century later by Legendre for the two cases only where the sum is *equal to* or *less than* two right angles.

The proofs are not free from imperfections, as when, in the proofs of Prop. XII. and the part of Prop. XIII. relating to the hypothesis of the *obtuse angle*, Saccheri uses Eucl. I. 18 depending on I. 16, a proposition which is only valid on the assumtion that *straight lines are infinite in length*; for this assumption itself does not hold under the hypothesis of the obtuse angle (the Riemann hypothesis).

The hypothesis of the acute angle takes Saccheri much longer to dispose of, and this part of the book is less satisfactory; but it contains the following propositions established anew by Lobachewsky and Bolyai, viz.:

(4) *Two straight lines in a plane* (even on the hypothesis of the acute angle) *either have a common perpendicular, or must, if produced in one and the same direction, either intersect once at a finite distance or at least continually approach one another.* (Prop. XXIII.)

(5) *In a cluster of rays issuing from a point there exist always* (on the hypothesis of the acute angle) *two determinate straight lines which separate the straight lines which intersect a fixed straight line from those which do not intersect it, ending with and including the straight line which has a common perpendicular with the fixed straight line.* (Props. XXX., XXXI., XXXII.)

Lambert.

A dissertation by G.S. Klügel, *Conatuum praecipuorum theoriam parallelarum demonstrandi recensio* (1763), contained an examination of some thirty "demonstrations" of Post. 5 and is remarkable for its conclusion expressing, apparently for the first time, *doubt as to its demonstrability* and observing that the certainty which we have in us of the truth of the Euclidean hypothesis is not the result of a series of rigorous deductions but rather of experimental observations. It also had the greater merit that it called the attention of Johann Heinrich Lambert (1728–1777) to the theory of parallels. His *Theory of Parallels* was written in 1766 and published after his death by G. Bernoulli and C.F. Hindenburg; it is reproduced by Engel and Stäckel (*op. cit.* pp. 152–208).

The third part of Lambert's tract is devoted to the discussion of the same three hypotheses as Saccheri's, the hypothesis of the *right angle* being for Lambert the *first*, that of the *obtuse angle* the *second*, and that of the *acute angle* the *third*, hypothesis; and, with reference to a quadrilateral with *three right angles* from which Lambert starts (that is, one of the halves into which the median divides Saccheri's quadrilateral), the three hypotheses are the assumptions that the fourth angle is a right angle, an obtuse angle, or an acute angle respectively.

Lambert goes much further than Saccheri in the deduction of new propositions from the *second* and *third* hypotheses. The most remarkable is the following.

The area of a plane triangle, under the second *and* third *hypotheses, is proportional to the difference betweeen the sum of the three angles and two right angles.*

Thus the numerical expression for the area of a triangle is, under the *third* hypothesis

$$\Delta = k(\pi - A - B - C) \quad \ldots\ldots \quad (1),$$

and under the *second* hypothesis

$$\Delta = k(A + B + C - \pi) \quad \ldots\ldots \quad (2),$$

where k is a positive constant.

A remarkable observation is appended (§ 82): "In connexion with this it seems to be remarkable that the *second* hypothesis holds if *spherical* instead of plane triangles are taken, because in the former also the sum of the angles is greater than two right angles, and the excess is proportional to the area of the triangle.

"It appears still more remarkable that what I here assert of spherical triangles can be proved independently of the difficulty of parallels."

This discovery that the *second* hypothesis is realised on the surface of a sphere is important in view of the development, later, of the Riemann hypothesis (1854).

Still more remarkable is the following prophetic sentence: "*I am almost inclined to draw the conclusion that the* third *hypothesis arises with an imaginary spherical surface*" (cf. Lobachewsky's *Géométrie imaginaire*, 1837).

No doubt Lamber was confirmed in this by the fact that, in the formula (2) above, which for $k = r^2$ represents the area of a spherical triangle, if $r\sqrt{-1}$ is substituted for r, and $r^2 = k$, we obtain the formula (1).

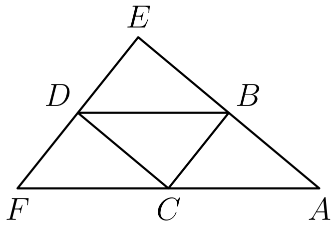
Legendre.

No account of our present subject would be complete without a full reference to what is of permanent value in the investigations of Adrien Marie Legendre (1752–1833) relating to the theory of parallels, which extended over the space of a generation. His different attempts to prove the Euclidean hypothesis appeared in the successive editions of his *Élements de Géométrie* from the first (1794) to the twelfth (1823), which last may be said to contain his last word on the subject. Later, in 1833, he published, in the *Mémoires de l'Académie Royale des Sciences*, XII. p. 367 sqq., a collection of his different proofs under the title *Réflexions sur différentes manières de démontrer la théorie des parallèles.* His exposition brought out clearly, as Saccheri had done, and kept steadily in view, the essential connexion between the theory of parallels and the sum of the angles of a triangle. In the first edition of the *Élements* the proposition that *the sum of the angles of a triangle is equal to two right angles* was proved analytically on the basis of the assumption that the choice of a *unit of length* does not affect the correctness of the proposition to be proved, which is of course equivalent to Wallis' assumption of *the existence of similar figures.* A similar analytical proof is given in the notes to the twelfth edition. In his second edition Legendre proved Postulate 5 by means of the assumption that, *given three points not in a straight line, there exists a circle passing through all three.* In the third edition (1800) he gave the proposition that *the sum of the angles of a trinagle is not greater than two right angles*; this proof, which was geometrical, was replaced later by another, the best known, depending on a construction like that of Euclid I. 16, the continued application of which enables any number of successive triangles to be evolved in which, while the sum of the angles in each remains always equal to the sum of the angles of the original triangle, one of the angles increases and the sum of the other two diminishes continually. But Legendre found the proof of the equally necessary proposition that the sum of the angles of a triangle is *not less* than two right angles to present great difficulties. He first observed that, as in the case of spherical triangles (in which the sum of the angles is greater than two right angles) the excess of the sum of the angles over two right angles is proportional to the area of the triangle, so in the case of rectilineal triangles, if the sum of the angles is less than two right angles by a certain *deficit*, the *deficit* will be proportional to the area of the triangle. Hence if, starting from a given triangle, we could construct another triangle in which the original triangle is contained at least

m times, the *deficit* of this new triangle will be equal to at least m times that of the original triangle, so that the sum of the angles of the greater triangles will diminish progressively as m increases, until it becomes zero or negative: which is absurd. The whole difficulty was thus reduced to that of the construction of a triangle containing the given triangle at least twice; but the solution of even this simple problem requires it to be assumed (or proved) that *through a given point within a given angle less than two-thirds of a right angle, we can always draw a straight line which shall meet both sides of the angle*. The proof in the course of which the necessity for the assumption appeared is as follows.

It is required to prove that the sum of the angles of a triangle cannot be *less* than two right angles.

Suppose A is the least of the three angles of a triangle ABC. Apply to the opposite side BC a triangle DBC, equal to the triangle ACB, and such that the angle DBC is equal to the angle ACB, and the angle DCB to the angle ABC; and *draw any straight line through D cutting AB, AC produced in E, F.*



If now the sum of the angles of the triangle ABC is less than two right angles, being equal to $2R-\delta$, say, the sum of the angles of the triangle DBC, equal to the triangle ABC, is also $2R-\delta$.

Since the sum of the three angles of the remaining triangles DEB, FDC respectively cannot at all events be *greater* than two right angles [for Legendre's proofs of this see below], the sum of the twelve angles of the four triangles in the figure *cannot be greater* than

$$4R + (2R - \delta) + (2R - \delta), \quad \text{i.e.} \quad 8R - 2\delta.$$

Now the sum of the three angles at each of the points B, C, D is $2R$. Subtracting these nine angles, we have the result that the three angles of the triangle AEF *cannot be greater* than $2R - 2\delta$.

Hence, if the sum of the angles of the triangle ABC is less than two right angles by δ, the sum of the angles of the larger triangle AEF is less than two right angles by *at least* 2δ.

We can continue the construction, making a still larger triangle from AEF, and so on.

But, however small δ is, we can arrive at a multiple $2^n\delta$ which shall exceed any given angle and therefore $2R$ itselff; so that the sum of the three angles of a triangle sufficiently large would be zero or even less than zero: which is absurd.

Therefore etc.

The difficultly caused by the necessity of making the above-mentioned assumption made Legendre abandon, in his ninth edition, the method of the editions from the third to the eighth and return to Euclid's method pure and simple.

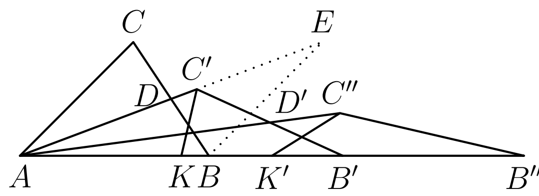
But again, in the twelfth, he returned to the plan of constructing any number of successive triangles such that the sum of the three angles in all of them remains equal to the sum of the three angles of the original triangle, but two of the angles of the new triangles become smaller and smaller, while the third becomes larger and larger; and this time he claims to prove in one proposition that the sum of the three angles of the original triangle is *equal* to two right angles by continuing the construction of new triangles *indefinitely* and compressing the two smaller angles of the ultimate triangle into nothing, while the third angle is made to become a *flat* angle at the same time. The construction and attempted proof are as follows.

Let ABC be the given triangle; let AB be the greatest side and BC the least; therefore C is the greatest angle and A the least.

From A draw AD to the middle point of BC, and produce AD to C', making AC' equal to AB.

Produce AB to B', making AB' equal to twice AD.

The triangle $AB'C'$ is then such that the sum of its three angles is equal to the sum of the three angles of the triangle ABC.



For take AK along AB equal to AD, and join $C'K$.

Then the triangles ABD, $AC'K$ have two sides and the included angles respectively equal, and are therefore equal in all respects; and $C'K$ is equal to BD or DC.

Next, in the triangles $B'C'K$, ACD, the angles $B'KC'$, ADC are equal, being respectively supplementary to the equal angles AKC', ADB; and the two sides about the equal angles are respectively equal; therefore the triangles $B'C'K$, ACD are equal in all respects.

Thus the angle $AC'B'$ is the sum of two angles respectively equal to the angles B, C of the original triangle; and the angle A in the original triangle is the sum of two angles respectively equal to the angles at A and B' in the triangle $AB'C'$.

It follows that the sum of the three angles of the new triangle $AB'C'$ is equal to the sum of the angles of the triangle ABC.

Moreover, the side AC', being equal to AB, and therefore greater than AC, is greater than $B'C'$ which is equal to AC.

Hence the angle $C'AB'$ is less than the angle $AB'C'$; so that the angle $C'AB'$ is less than $\frac{1}{2}A$, where A denotes the angle CAB of the original triangle.

[It will be observed that the triangle $AB'C'$ is really the same triangle as the triangle AEB obtained by the construction of Eucl. I. 16, but differently placed so that the longest side lies along AB.]

But taking the middle point D' of the side $B'C'$ and repeating the same construction, we obtain a triangle $AB''C''$ such that (1) the sum of its three angles is equal to the sum of the three angles of ABC, (2) the sum of the two angles $C''AB''$, $AB''C''$ is equal to the angle $C'AB'$ in the preceding triangle, and is therefore less than $\frac{1}{2}A$, and (3) the angle $C''AB''$ is less than half the angle $C'AB'$, and therefore less than $\frac{1}{4}A$.

Continuing in this way, we shall obtain a triangle Abc such that the sum of two angles, those at A and b, is less than $\dfrac{1}{2^n}A$, and the angle at c is greater than the corresponding angle in the preceding triangle.

If, Legendre argues, the construction be continued indefinitely so that $\dfrac{1}{2^n}A$ becomes smaller than any assigned angle, the point c ultimately lies on Ab, and the sum of the three angles of the triangle (which is equal to the sum of the three angles of the original triangle) becomes identical with the angle at c, which is then a *flat* angle, and therefore equal to two right angles.

This proof was however shown to be unsound (in respect of the final inference) by J. P. W. Stein in Gergonne's *Annales de Mathématiques* XV., 1824, pp. 77–79.

We will now reproduce shortly the substance of the theorems of Legendre which are of the most permanent value as not depending on a particular hypothesis as regards parallels.

I. *The sum of the three angles of a triangle cannot be* greater *than two right angles.*

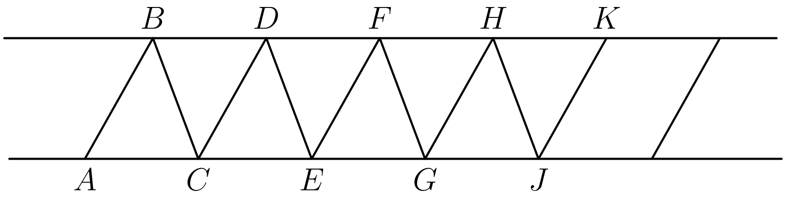
This Legendre proved in two ways.

(1) *First proof* (in the third edition of the Éléments).

Let ABC be the given triangle, and ACJ a straight line.

Make CE equal to AC, the angle DCE equal to the angle BAC, and DC equal to AB. Join DE.

Then the triangle DCE is equal to the triangle BAC in all respects.



If then the sum of the three angles of the triangle ABC is greater than $2R$, the said sum must be greater than the sum of the angles BCA, BCD, DCE, which sum is *equal* to $2R$.

Subtracting the equal angles on both sides, we have the result that the angle ABC is *greater* than the angle BCD.

But the two sides AB, BC of the triangle ABC are respectively equal to the two sides DC, CB of the triangle BCD.

Therefore the base AC is *greater* than the base BD (Eucl. I. 24).

Next, make the triangle FEG (by the same construction) equal in all respects to the triangle BAC or DCE; and we prove in the same way that CE (or AC) is *greater* than DF.

And, at the same time, BD is equal to DF, because the angles BCD, DEF are equal.

Continuing the construction of further triangles, however small the difference between AC and BD is, we shall ultimately reach some multiple of this difference, represented in the figure by (say) the difference between the straight line AJ and the composite line $BDFKH$, which will be greater than any assigned length, and greater therefore than the sum of AB and JK.

Hence, on the assumption that the sum of the angles of the triangle ABC is greater than $2R$, the broken line $ABDFHKJ$ may be less than the straight line AJ: which is impossible.

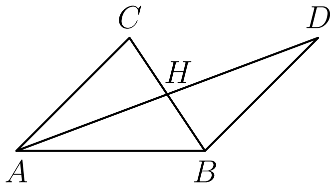
Therefore etc.

(2) *Proof substituted later.*

If possible, let $2R+\alpha$ be the sum of the three anglse of the triangle ABC, of which A is not greater than either of the others.

Bisect BC at H, and produce AH to D, making HD equal to AH; join BD.

Then the triangles AHC, DHB are equal in all respects (I. 4); and the angles CAH, ACH are respectively equal to the angles BDH, DBH.



It follows that the sum of the angle of the triangle ABD is equal to the sum of the angles of the original triangle, i.e. to $2R + \alpha$.

And one of the angles DAB, ADB is either equal to or less than half the angle CAB. Proceeding in this way, we arrive at a triangle in which the sum of the angles is $2R + \alpha$, and only of them is not greater than $(\angle CAB)/2^n$.

And if n is sufficiently large, this will be less than α; in which case we should have a triangle in which two angles are together greater than two right angles: which is absurd.

Therefore α is equal to or less than zero.

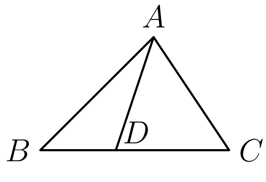
(It will be noted that in both these proofs, as in Euclid. I. 16, it is taken for granted that *a straight line is infinite in length* and does not return into itself, which is not true under the Riemann hypothesis.)

II. On the assumption that the sum of the angles of a triangle is *less* than two right angles, *if a triangle is made up of two others, the "deficit" of the former is equal to the sum of the "deficits" of the others..*

In fact, if the sums of the angles of the component triangles are $2R - \alpha$, $2R - \beta$ respectively, the sum of the angles of the whole triangle is

$$(2R - \alpha) + (2R - \beta) - 2R = 2R - (\alpha + \beta).$$

III. *If the sum of the three angles of a triangle is* equal *to two right angles, the same is true of all triangles obtained by subdividing it by straight lines drawn from a vertex to meet the opposite side.*



Since the sum of the angles of the triangle ABC is equal to $2R$, if the sum of the angles of the triangle ABD were $2R - \alpha$, it would follow that the sum of the angles of the triangle ADC must be $2R + \alpha$, which is absurd by I. above).

IV. *If in a triangle the sum of the three angles is* equal *to two right angles, a quadrilateral can always be constructed with four right angles and four equal sides exceeding in length any assigned rectilineal segment.*

Let ABC be a triangle in which the sum of the angles is equal to two right angles. We can assume ABC to be an *isosceles right-angled* triangle because we can reduce the case to this by making subdivisions of ABC by straight lines through vertices (as in Prop. III. above).

Taking two equal triangles of this kind and placing their hypotenuses together, we obtain a quadrilateral with four right angles and four equal sides.

Putting four of these quadrilaterals together, we obtain a new quadrilateral of the same kind but with its sides double of those of the first quadrilateral.

After n such operations we have a quadrilateral with four right angles and four equal sides, each being equal to 2^n times the side AB.

The diagonal of this quadrilateral divides it into two equal isosceles right-angled triangles in each of which the sum of the angles is equal to two right angles.

Consequently, from the existence of *one* triangle in which the sum of the three angles is equal to two right angles, if follows that there exists an isosceles right-angled triangle with sides greater than any assigned rectilineal segment and such that the sum of its three angles is also equal to two right angles.

V. *If the sum of the three angles of* one *triagle is equal to two right angles, the sum of the three angles of* any other *triangle is also equal to two right angles.*

It is enough to prove this for a *right-angled* triangle, since any triangle can be divided into two right-angled triangles.

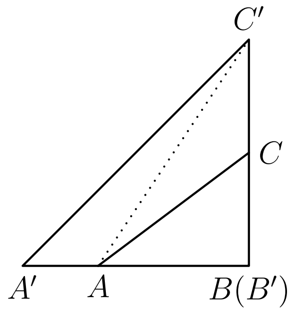
Let ABC be any right-angled triangle.

If then the sum of the angles of any one triangle is equal to two right angles, we can construct (by the preceding Prop.) an isosceles right-angled triangle with the same property and with its perpendicular sides greater than those of ABC.

Let $A'B'C'$ be such a triangle, and let it be applied to ABC as in the figure.

Applying then Prop. III. above, we deduce first that the sum of the three angles of the triangle $AB'C'$ is equal to two right angles, and next, for the same reason, that the sum of the three angles of the original triangle ABC is equal to two right angles.

If in any one *triangle the sum of the three angles is* less *than two right*



angles, the sum of the three angles of any other *triangle is also less than two right angles.*

This follows from the preceding theorem.

(It will be observed that the last two theorems are included amongst those of Saccheri, which contain however in addition the corresponding theorem touching the case where the sum of the angles is *greater* than two right angles.)

We come now the the bearing of these propositions upon Euclid's Postulate 5; and the next theorem is

VII. *If the sum of the three angles of a triangle is equal to two right angles, through any point in a plane there can only be drawn one parallel to a given straight line.*

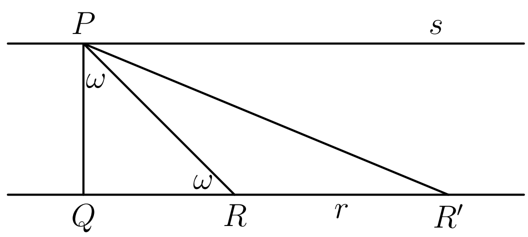
For the proof of this we require the following

Lemma. *It is always possible, through a point* P*, to draw a straight line which shall make, with a given straight line* (r)*, an angle less than any assigned angle.*

Let Q be the foot of the perpendicular from P upon r.

Let a segment QR be taken on r, on either side of Q, such that QR is equal to PQ.

Join PR, and mark off the segment RP' equal to PR; join PR'.



If ω represents the angle QPR or the angle QRP, each of the equal angles RPR', $RR'P$ is not greater than $\omega/2$.

Continuing the construction, we obtain, after the requisite number of operations, a triangle $PR_{n-1}R_n$ in which each of the equal angles is equal to or less than $\omega/2^n$.

Hence we shall arrive at a straight line PR_n which, starting from P and meeting r, makes with r an angle as small as we please.

To return now to the Proposition. Draw from P the straight line s perpendicular to PQ.

Then any straight line drawn from P which meets r in R will form equal angles with r and s, since, by hypothesis, the sum of the angles of the triangle PQR is equal to two right angles.

And since, by the Lemma, it is always possible to draw through P straight lines which form with r angles as small as we please, it follows that all the straight lines through P, except s will meet r. Hence s is the only parallel to r that can be drawn through P.

The history of the attempts to prove Postulate 5 or something equivalent has now been brought down to the parting of the ways. The further developments on lines independent of the Postulate, beginning with Schweikart (1780–1857), Taurinus (1794–1874), Gauss (1777-1855), Lobachewsky (1793–1856), J. Bolyai (1802–1860), Riemann (1826–1866), belong to the history of non-Euclidean geometry, which is outside the scope of this work. I may refer the reader to the full article *Sulla teoria delle parallele e sulle geometrie non-euclidee* by R. Bonola in *Questioni riguardanti le mathematiche elementari*, I., of which I have made considerable use in the above, to the same author's *La geometria non-euclidea*, Bologna, 1906, to the first volume of Killing's *Einführung in die Grundlagen der Geometrie*, Paderborn, 1893, to P. Mansion's *Premiers principes de métagéometrie*, and P. Barbarin's *La géometrie non-Euclidienne*, Paris, 1902, to the historical summary in Veronese's *Fondamenti di geometria*, 1891, p. 565 sqq., and (for original sources) to Engel and Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, 1895, and *Urkunden zur Geschichte der nicht-Euklidischen Geometrie*, I. (Lobachewsky), 1899, and II. (Wolfgang und Johann Bolyai). I will only add that it was Gauss who first expressed a conviction that the Postulate could never be proved; he indicated this in reviews in the *Göttingische gelehrte Anzeigen*, 20 Apr. 1816 and 28 Oct. 1822, and affirmed it in a letter to Bessel of 27 January, 1829. The actual indemonstrability of the Postulate was proved by Beltrami (1868) and by Hoüel (*Note sur l'impossibilité de démontrer par une construction plane le principe de la théorie des parallèles dit Postulatum d'Euclide* in Battaglini's *Giornale di matematiche*, VIII., 1870, pp. 84–89).

Alternatives for Postulate 5.

It may be convenient to collect here a few of the more noteworthy substitutes which have from time to time been formally suggested or tacitly assumed.

(1) *Through a given point only one parallel can be drawn to a given straight line* or, *Two straight lines which intersect one another cannot both be parallel to one and the same straight line.*

This is commonly known as "Playfair's Axiom," but it was of course not a new discovery. It is distinctly stated in Proclus' note to Eucl. I. 31.

(1 *a*) *If a straight line intersect one of two parallels, it will intersect the other also* (Proclus).

(1 *b*) *Straight lines parallel to the same straight line are parallel to one another.*

The forms (1 *a*) and (1 *b*) are exactly equivalent to (1).

(2) *There exist straight lines everywhere equidistant from one another* (Posidonus and Geminus); with which may be compared Proclus' tacit assumption that *Parallels remain, throughout their length, at a finite distance from one another.*

(3) *There exists a triangle in which the sum of the three angles is equal to two right angles* (Legendre).

(4) *Given any figure, there exists a figure similar to it of any size we please* (Wallis, Carnot, Laplace).

Saccheri points out that it is not necessary to assume so much, and that it is enough to postulate that *there exist two unequal triangles with equal angles.*

(5) *Through any point within an angle less than two-thirds of a right angle a straight line can always be drawn which meets both sides of the angle* (Legendre).

With this may be compared the similar axiom of Lorenz (*Grundriss der reinen und andgewandten Mathematik*, 1791): *Every straight line through a point within an angle must meet one of the sides of the angle.*

(6) *Given any three points not in a straight line, there exists a circle passing through them* (Legendre, W. Bolyai).

(7) *"If I could prove that a rectilineal triangle is possible the content of which is greater than any given area, I am in a position to prove perfectly rigorously the whole of geometry"* (Gauss, in a letter to W. Bolyai, 1799).

Cf. the proposition of Legendre numbered IV. above, and the axiom of Worpitzky: *There exists no triangle in which every angle is as small as we please.*

(8) *If in a quadrilateral three angles are right angles, the fourth angel is a right angle also* (Clairaut, 1741).

(9) *if two straight lines are parallel, they are figures opposite to (or the reflex of) one another with respect to the middle points of all their transversal segments* (Veronese, *Elementi*, 1904).

Or, *two parallel straight lines intercept, on every transversal which passes through the middle point of a segment included between them, another segment the middle point of which is the middle point of the first* (Ingrami, *Elementi*, 1904).

Veronese and Ingrami deduce immediately Playfair's Axiom.