[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 200–201 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Postulate 4.]

Postulate 4.

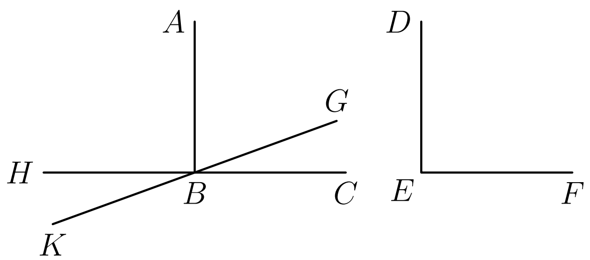
Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις ειναι.

That all right angles are equal to one another.

While this Postulate asserts the essential truth that a right angle is a *determinate magnitude* so that it really serves as an invariable standard by which other (acute and obtuse) angles may be measured, much more than this is implied, as will easily be seen from the following consideration. If the statement is to be *proved*, it can only be proved by the method of applying one pair of right angles to another and so arguing their equality. But this method would not be valid unless on the assumption of the *invariability of figures*, which would therefore have to be asserted as an antecedent postulate. Euclid preferred to assert as a postulate, directly, the fact that all right angles are equal; and hence his postulate must be taken as equivalent to the principle of *invariability of figures* or its equivalent, the *homogeneity of space*.

According to Proclus, Geminus held that this Postulate should not be classed as a postulate but as an axiom, since it does not, like the first three Postulates, assert the possibility of some *construction* but expresses an essential property of right angles. Proclus further observed (p. 188, 8) that it is not a postulate in Aristotle's sense either. (In this I think he is wrong, as explained above.) Proclus himself, while regarding the assumption as axiomatic ("the equality of right angles suggests itself even by virtue of our common notions"), is prepared with a proof, if such is asked for.

Let ABC, DEF be two right angles. If they are not equal, one of them must be the greater, say ABC.



Then, if we apply DE to AB, EF will fall within ABC, as BG. Produce CB to H. Then, since ABC is a right angle, so is ABH, and the two angles are equal (a right angle being by definition equal to its adjacent angle).

Therefore the angle GBH is *greater* than the angle ABG.

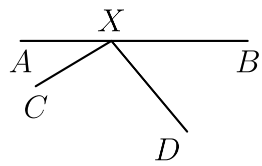
Producing GB to K, we have similarly the two angles ABK, ABG both right and equal to one another; whence the angle ABH is *less* than the angle ABG.

But it is also greater: which is impossible.

Therefore etc.

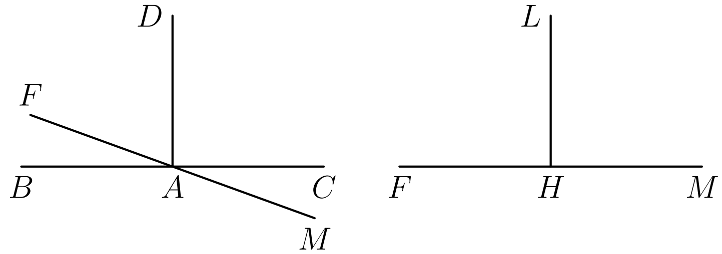
A defect in this proof is the assumption that CB, GB can each be produced only in one way, and the BK falls outside the angle ABH.

Saccheri's proof is more careful in that he premises a third lemma in addition to those asserting (1) that two straight lines cannot enclose a space and (2) that two straight lines cannot have a common segment. The third lemma is: *If two straight lines AB, CXD meet one another at an intermediate point X, they do not* touch *at that point, but* cut one another.



Suppose now that DA standing on BAC makes the two angles DAB, DAC equal, so that each is a right angle by the definition.

Similarly, let LH form with the straight line FHM the right angles LHF, LHM. Let DA, HL be equal; and suppose the two of the second figure so laid upon the first that the point H falls on A, and L on D.



Then the straight line FHM will (by the third lemma) not *touch* the straight line BC at A; it will either

(a) coincide exactly exactly with BC, or

(b) *cut* it so that one of its extremities, as F, will fall above [BC] and the other, M, below it.

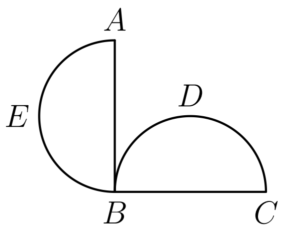
If the alternative (a) is true, we have already prove the exact equality of all rectilineal right angles.

Under alternative (b) we prove that the angle LHF, being equal to the angle DAF, is less than the angle DAB or DAC, and *a fortiori* less than the angle DAM or LHM: which is contrary to the hypothesis.

[Hence (a) is the only possible alternative, so that all right angles are equal.]

Saccheri adds that it makes no difference if the angle DAF diverges *infinitely little* from the angle DAB. This would equally lead to a contradiction contradicting the hypothesis.

It will be observed that Saccheri speaks of "the exact equality of all *rectilineal* right angles." He may have had in mind the remark of Pappus, quoted by Proclus (p. 189, 11), that the converse of this postulate, namely that an angle which is equal to a right angle is also right, is not necessarily true, unless the former angle is *rectilineal.* Suppose two equal straight lines BA, BC at right angles to one another, and semi-circles described on BA, BC respectively as AEB, BDC in the figure. Then, since the semi-circles are equal, they coincide if applied to one another. Hence the "angles" EBA, DBC are equal. Add to each the "angle" ABD; and it follows that the *lunular* angle EBD is equal to the right angle ABC. (Similarly, if BA, BC be inclined at an acute or obtuse angle, instead of at a right angle, we find a *lunular* angle equal to an acute or obtuse angle.) This is one of the curiosities which Greek commentators delighted in.



Veronese, Ingrami, and Enriques and Amaldi deduce the fact that *all right angles are equal* from the equivalent fact that *all flat angles are equal*, which is either itself assumed as a postulate or immediately deduced from some other postulate.

Hilbert takes quite a different line. He considers that Euclid did wrong in placing Post. 4 among "axioms." He himself, after his Group III. of Axioms containing six relating to congruence, proves several theorems about the congruence of triangles and angles, and then deduces our Postulate.

As to the *raison d'être* and the place of Post. 4 one thing is quite certain. If was essential from Euclid's point of view that it should come before Post. 5, since the condition in the latter that a certain pair of angle are together less than two right angles would be useless unless it were first made clear that right angles are angles of determinate and invariable magnitude.