[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 196–199 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Postulate 2.]

Postulate 2.

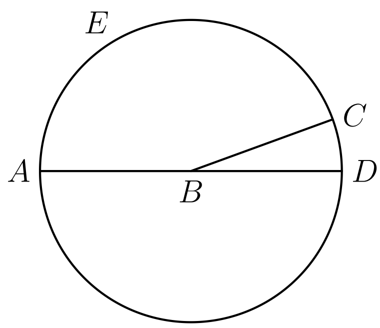
Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

To produce a finite straight line continuously in a straight line.

I translate πεπερασμένην by *finite*, because that is the received equivalent, and because any alternative word such as *limited, terminated*, if applied to a straight line, would equally fail to express what modern Italian geometers aptly call a *rectilineal segment*, that is, a straight line having *two* extremities.

Just as Post. 1 asserting the possibility of drawing a straight line form any one point to another must be held to declare at the same time that the straight line so drawn is unique, so Post. 2 maintaining the possibility of producing a finite straight line (a "rectilineal segment") continuously in a straight line must also be held to assert that the striaght line can only be produced *in one way* at either end, or that the produced part in either direction is *unique*; in other words, that *two straight lines cannot have a common segment*. This latter assumption is not expressly appealed to by Euclid until XI. 1. But it is needed at the very beginning of Book 1. Proclus (p. 214, 18) says that Zeno of Sidon, an Epicurean, maintained that the very first proposition I. 1 requires it to eb admitted that "two straight lines cannot have the same segments"; otherwise AC, BC might meet before they arrive at C and have the rest of their length common, in which case the actual triangle formed by them and AB would not be equilateral. The assumption that two straight lines cannot have a common segment is certainly necessary in I. 4, where one side of one triangle is placed on that side of the other triangle which is equal to it, and it is inferred that the two coincide throughout their length: this would by no means follow if two straight lines could have a common segment. Proclus (p. 215, 24), while observing that Post. 2 clearly indicates that the produced portion must be *one*, attempts to prove it, but unsuccessfully. Both he and Simplicius practically use the same argument. Suppose, says Proclus, that the straight lines AC, AD have AB as a common segment. With centre B and radius BA describe a circle (Post. 3) meeting AC, AD in C, D. Then, since ABC is a straight line through the centre, AEC is a semi-circle. Similarly ABD being a straight line through the centre, AED is a semi-circle. Therefore AEC is equal to AED: which is impossible.

Proclus observes that Zeno would object to this proof as really depending on the assumption that "two circumferences (of circles) cannot have one



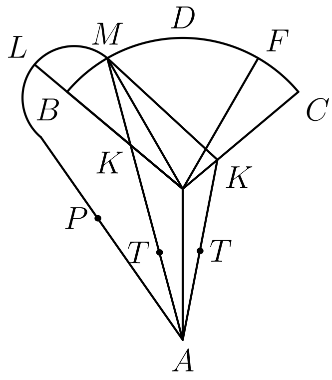
portion common"; for this, he would say, is assumed in the common proof by superposition of the fact that a circle is bisected by a diameter, since that proof takes it for granted that, if one part of the circumference cut off by the diameter, when applied to the other, does not coincide with it, it must necessarily fall either *entirely* outsidef or *entirely* inside it, whereas there is nothing to prevent their coinciding, not altogether, but in part only; and, until you really prove the bisection of a circle by its diameter, the above proof is not valid. Posidonius is represented as having derided Zeno for not seeing that the proof of the bisection of a circle by its diameter goes on just as well if the circumferences fail to coincide *in part* only. But the true objection to the proof above given is that the proof of the bisection of the circle by any diameter *itself* assumes that two straight ines cannot have a common segment; for, if we wish to draw the diameter of a circle which has its extremity at a given point of the circumference we have to join the latter point to the centre (Post. 1) and then to *produce* the straight line so drawn till it meets the centre again (Post. 2), and it is necessary for the proof that the produced part shall be *unique*.

Saccheri adopted the proper order when he gave, first the proposition that two straight lines cannot have a common segment, and after that the proposition that any diameter of a circle bisects the circle and its circumference.

Saccheri's proof of the former is very interesting as showing the thoroughness of his method, if not at the end entirely convincing. It is in five stages which I shall indicate shortly, giving the full argument of the first only.

Suppose, if possible, hat AX is a common segment of both the straight lines AXB, AXC, in one plane, produced beyond X. Then describe about X as centre, with radius XB or XC, the arc BMC, and draw through X to any point on it the straight line XM.

(i) I maintain that, with the assumption made, *the line AXM is also a straight line which is drawn from the point* A *to the point* X *and produced beyond* X.



For if this line were not straight, we could draw another straight line AM which for its part would be straight. This straight line will either (a) cut one of the two straight lines XB, XC in a certain point K or (b) enclose one of them, for instance XB, in the area bounded by AX, XM and $APLM$.

But the first alternative (a) obviously contradicts the foregoing lemma [that two straight lines cannot enclose a space], since in that case the two lines AXK, ATK, which by hypothesis are straight, would enclose a space.

The second possibility (b) is at once seen to involve a similar absurdity. For the straight line XB must, when produced beyond B, ultimately meet $APLM$ in a point L. Consequently the two lines $AXBL$, APL, which by hypothesis are straight, would again enclose a space. If however we were to assume that the straight line XB produced beyond B will ultimately meet either the straight line XM or the straight line XA in another point, we should in the same way arrive at a contradiction.

From this it obviously follows that, on the assumption made, the line AXM is itself the straight line which was drawn from the point A; and that is what was maintained.

The remaining stages are in substance these.

(ii) *If the straight line AXB, regarded as rigid, revolves about* AX *as axis, it cannot assume* two *more positions in the same plane, so that, for example, in one position* XB *should coincide with* XC*, and in the other with* XM*.*

[This is proved by considerations of symmetry. AXB cannot be altogether "similar or equal to" AXC, if viewed from the same side (left or right) of both: otherwise they would coincide, which by hypothesis they do not. But there is nothing to prevent AXB viewed from one side (say the left) being "similar or equal to" AXC viewed from the other side (i.e. the right), so that AXB *can*, without any change, be brought into the position AXC.

AXB cannot however take the position of the other striaght line AXM as well. If they were like on one side, they would coincide; if they were like on opposite sides, AXM, AXC would be like on the same side and therefore coincide.]

(iii) The other positions of AXB during the revolution must be above or below the original plane.

(iv) It is next maintained that *there* is *a point* D *on the arc* BC *such that, if* XD *is drawn,* AXD *is not only a straight line but is such that viewed from the left side it is exactly "similar or equal" to what it is when viewed from the right side.*

[*First*, it is proved that points M, F can be found still nearer together, and so on continually, until either (a) we come to *one* point D such that *AXD is exactly like itself when the right and left sides are compared* or (b) there are *two* ultimate points of this sort M, F, so that *both* AXM*,* AXF *have this property.*

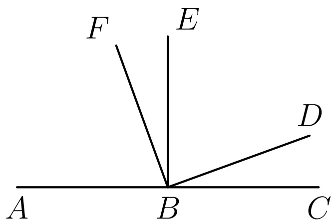
Thirdly, (b) is ruled out by reference to the definition of a straight line. Hence (a) only is true, and there is only *one* point D such as described.]

(v) Lastly, Saccheri concludes that the straight line AXD so determined "is *alone* a straight line, and the *immediate* prolongation from A beyond X to D," relying again on the definition of a straight line as "lying evenly."

Simson deduced the proposition that *two straight lines cannot have a common segment* as a corollary from I. 11; but his argument is a complete *petitio principii*, as shown by Todhunter in his note on that proposition.

Proclus (p. 217, 10) records an ancient proof also based on the proposition I. 11. Zeno, he says, propounded this proof and then criticised it.

Suppose that two straight lines AC, AD have a common segment AB, and let BE be drawn at right angles to AC.



Then the angle EBC is right.

If then the angle EBD is also right, the two angles will be equal: which is impossible.

If the angle EBD is not right, draw BF at right angles to AD; therefore the angle FBA is right.

But the angle EBA is right.

Therefore the angles EBA, FBA are equal: which is impossible.

Zeno objected to this, says Proclus, because it assumed the later proposition I. 11 for its proof. Posidonius said that there was no trace of such a proof to be found in the text-books of the Elements, and that it was only invented by Zeno for the purpose of slandering contemporary geometers. Posidonius maintains further that even this proof has something to be said for it. There must be some straight line at right angles to each of the two straight lines AC, AD (the very definition of right angles assumes this): "*suppose then it appears to be the straight line we have set up.*" Here then we have an ancient instance of a defence of *hypothetical construction*, but in such apologetic terms ("it is possible to say *something* even for this proof") that we may conclude that in general it would not have been accepted by geometers of that time as a legitimate means of proving a proposition.

Todhunter proposed to deduce that *two straight lines cannot have a common segment* from I. 13. But this will not serve either, since, a before mentioned, the assumption is really required for I. 4.

It is best to make it a postulate.