[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 195–196 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Postulate 1.]

Postulate 1.

Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

Let the following be postulated: to draw a straight line from any point to any point.

From any point to any point. In general statements of this kind the Greeks did not say, as we do, "*any* point," "*any* triangle" etc., but "*every* point," "*every* triangle" and the like. Thus the words here are literally "from every point to every point." Similarly the first words of Postulate 3 are "with *every* centre and distance," and the enunciation, e.g., of I. 18 is "In *every* triangle the greater side subtends the greater angle."

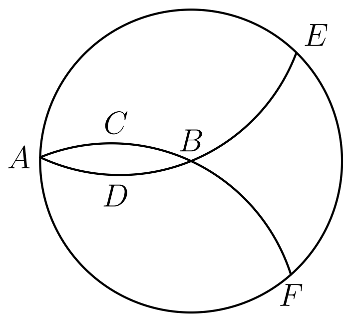
It will be remembered that, according to Aristotle, the geometer must in general assume *what* a thing is, or its definition, but must prove *that* it is, i.e. the *existence* of the thing corresponding to the definition: only in the case of the two most primary things, points and lines, does he assume, without proof, both the definition and the existence of the thing defined. Euclid has indeed no separate assumption affirming the existence of *points* such as we find nowadays in text-books like those of Veronese, Ingrami, Enriques, "there exist distinct points" or "there exist an infinite number of points." But, as regards the only lines dealt with in the *Elements*, straight lines and circles, existence is asserted in Postulates 1 and 3 respectively. Postulate 1 however does much more than (1) postulate the existence of straight lines. It is (2) an answer to a possible objector who should say that you cannot, with the imperfect instruments at your disposal, draw a mathematical straight line at all, and consequently (in the words of Aristotle, *Anal. post.* I. 10, 76 b 41) that the geometer uses false hypotheses, since he calls a line a foot long when it is not or straight when it is not straight. It would seem (if Gherard's translation is right) that an-Nairīsī saw that one purpose of the Postulate was to refute this criticism: "the utility of the first three postulates is (to ensure) that the weakness of our equipment shall not prevent (scientific) demonstration" (ed. Curtze, p. 30). The fact is, as Aristotle says, that the geometer's demonstration is not concerned with the particular imperfect straight line which he has drawn, but with the ideal straight line of which it is the imperfect representation. Simplicius too indicates that the object of the Postulate is rather to enable the drawing of a mathematical straight line to be *imagined* than to assert that it can actually be realised in practice:

"he would be a rash person who, taking things as they actually are, should postulate the drawing of a straight line from Aries to Libra."

There is still something more that must be inferred from the Postulate combined with the definition of a straight line, namely (3) that the straight line joining two points is *unique*: in other words that, *if two straight lines* ("rectilineal segments," as Veronese would call them) *have the same etremities, they must coincide through out their length.* The omission of Euclid to state this in so many words, though he assumes it in I. 4, is no doubt answerable for the interpolation in the text of the equivalent assumption that *two straight lines cannot enclose a space*, which has consequently appeared in MSS. and editions of Euclid, either amongh Axioms or Postulates. That Postulate 1 included it, by conscious implication, is even clear from Proclus's words in his note on I. 4 (p. 239, 16): "therefore two straight lines do not enclose a space, and it was with knowledge of this fact that the writer of the Elements said in the first of his Postulates, *to draw a straight line from any point to any point*, implying that it is *one* straight line which would always join the two points, not *two*."

Proclus attempts in the same note (p. 239) to *prove* that two straight lines cannot enclose a space, using as his basis the definition of the diameter of a circle and the theorem, stated in it, that any diameter divides the circle into two equal parts.

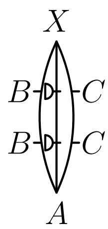
Suppose, he says, ACB, ADB to be two straight lines enclosing a space. Produce them (beyond B) indefinitely. With centre B and distance AB describe a circle, cutting the lines so produced in F, E respectively.



Then, since $ACBF$, $ADBE$ are both diameters cutting off semi-circles, the arcs AE, AEF are equal: which is impossible. Therefore etc.

It will be observed, however, that the straight lines produced are assumed to meet the circle given in two *different* points E, F, whereas, for anything we know, E, F might coincide and the straight lines have *three* common points. The proof is therefore delusive.

Saccheri gives a different proof. From Euclid's definition of a straight line as that which lives evenly with its points he infers that, when such a line is turned about its two extremities, which remain fixed, all the points on it must remain throughout in the same position, and cannot take up different positions as the revolution proceeds. "In this view of the straight line the truth of the assertion that two straight lines do not enclose a space is obviously involved. In fact, if two lines are given which enclose a space, and of which the two points A and X are the common extremities, it is easily shown that neither, or else only one, of the two lines is straight."



It is however better to assume as a *postulate* the fact, inseparably connected with the idea of a straight line, that *there exists only one straight line containing two given points*, or, *if two straight lines have two points in common, they coincide throughout.*