[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 224–231 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Common Notion 4.]

Common Notion 4.

Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἄλληλα ἴσα ἀλλήλοις ἐστίν.

Things which coincide with one another are equal to one another.

The word ἐφαρμόζειν, as a geometrical term, has a different meaning according as it is used in the active or in the passive. In the passive, ἐφαρμόζεσθαι, it means "to be *applied* to" without any implication that the applied figure will exactly fit, or coincide with, the figure to which it is applied; on the other hand the active ἐφαρμόζειν is used intransitively and means "to fit exactly," "to coincide with." In Euclid and Archimedes ἐφαρμόζειν is constructed with ἐπί and the accusative, in Pappus with the dative.

On *Common Notion* 4 Tannery observes that it is incontestably geometrical in character, and should therefore have been excluded from the *Common Notions*; again, it is difficult to see why it is not accompanied by its converse, at all events for straight lines (and, it might be added, angles also), which Euclid makes use of in I. 4. As it is, says Tannery, we have here a definition of geometrical equality more or less sufficient, but not a real axiom.

It is true that Proclus seems to recognize this *Common Notion* and the next as proper axioms in the passage (p. 196, 15–21) where he says that we should not cut down the axioms to the minimum, as Heron does in giving only three axioms; but the statement seems to rest, not on authority, but upon an assumption that Euclid would state explicitly at the beginning all axioms subsequently used and not reducible to others unquestionably included. Now in I. 4 this *Common Notion* is not quoted; it is simply inferred that "the base BC will coincide with EF, *and will be equal* to it." The position is therefore the same as it is in regard to the statement in the same proposition that, "if... the base BC does not coincide with EF, *two straight lines will enclose a space*: which is impossible"; and, if we do not admit that Euclid had the axiom that "two straight lines cannot enclose a space," neither need we infer that he had *Common Notion* 4. I am therefore inclined to think that the latter is more likely than not to be an interpolation.

It seems clear that the Common Notion, as here formulated, is intended to assert that superposition is a legitimate way of proving the equality of two figures which have the necessary parts respectively equal, or, in other words, to serve as an *axiom of congruence*.

The phraseology of the propositions, e.g. I. 4 and I. 8, in which Euclid employs the method indicated, leaves no room for doubt that he regarded one figure as actually *moved* and *placed upon* the other. Thus in I. 4 he says, "The triangle ABC being applied (ἐφαρμοζομένου) to the triangle DEF, and the point A being *placed* (τιθεμένου) upon the point D, and the straight line AB on DE, the point B will also coincide with E, because AB is equal to DE"; and in I. 8, "If the sides BA, AC do not coincide with ED, DF, but *fall beside them* (take a different position, παραλλάξουσιν), then" etc. At the same time, it is clear that Euclid disliked the method and avoided it wherever he could, e.g., in I. 26, where he proves the equality of two triangles which have two angles equal to two angles and one side of the one equal to the corresponding side of the other. It looks as though he found the method handed down by tradition (we can hardly suppose that, if Thales proved that the diameter of a circle divides it into two equal parts, he would do so by any other method than that of superposition), and followed it, in the few cases where he does so, only because he had not been able to see his way to a satisfactory substitute. But seeing how much of the *Elements* depends on I. 4, directly or indirectly, the method can hardly be regarded as being, in Euclid, of only subordinate importance; on the contrary, it is fundamental. Nor, as a matter of fact, do we find in the ancient geometers any expression of doubt as to the legitimacy of the method. Archimedes uses it to prove that any spheroidal figure cut by a plane through the centre is divided into two equal parts in respect of both its surface and its volume; he also postulates in *Equilibrium of Planes* I. that "when equal and similar plane figures coincide if applied to one another, their centres of gravity coincide also."

Killing (*Einführung in die Grundlagen der Geometrie*, II. pp. 4, 5) contrasts the attitude of the Greek geometers with that of the philosophers, who, he says, appear to have agreed in banishing motion from geometry altogether. In support of this he refers to the view frequently expressed by Aristotle that mathematics has to do with *immovable* objects (ἀκίνητα), and that only where astronomy is admitted as part of mathematical science is motion mentioned as a subject for mathematics. Cf. *Metaph.* 989 b 32 "For mathematical objects are among things which exist apart from motion, except such as relate to astronomy"; *Metaph.* 1064 a 30 "Physics deals with things which have in themselves the principle of motion; mathematics is a theoretical science and one concerned with things which are *stationary* (μένοντα) but not separable" (sc. from matter); in *Physics* II. 2, 193 b 34 he speaks of the subjects of mathematics as "in thought separable from motion."

But I doubt whether in Aristotle's use of the words "immovable," "without motion" etc. as applied to the subjects of mathematics there is any implication such as Killing supposes. We arrive at mathematical concepts by abstraction from material objects; and just as we, in thought, eliminate the matter, so according to Aristotle we eliminate the attributes of matter as such, e.g. qualitative change and *motion.* It does not appear to me that the use of "immovable" in the passages referred to means more than this. I do not think that Aristotle would have regarded it as illegitimate to *move* a geometrical figure from one position to another; and I infer this from a passage in *De caelo* III. I where he is criticising "those who make up every body that has an origin by putting together *planes*, and resolve it again into *planes*." The reference must be to the *Timaeus* (54 B sqq.) where Plato evolves the four elements in this way. He begins with a right-angled triangle in which the hypotenuse is double of the smaller side; six of these put together in the proper way produce one equilateral triangle. Making solid angles with (a) three, (b) four, and (c) five of these equilateral triangles respectively, and taking the requisite number of these solid angles, namely four of (a), six of (b) and twelve of (c) respectively, and putting them together so as to form regular solids, he obtains (α) a tetrahedron, (β) an octahedron, (γ) an icosahedron respectively. For the fourth element (earth), four isosceles right-angled triangles are first put together so as to form a square, and then six of these squares are put together to form a cube. Now, says Aristotle (299 b 23), "it is absurd that planes should only admit of being put together so as to touch in a *line*; for just as a line and a line are put together in both ways, lengthwise and breadthwise, so must a plane and a plane. A line can be combined with a line in the sense of being a line *superposed*, and not *added*"; the inference being that a *plane* can be superposed on a *plane.* Now this is precisely the sort of motion in question here; and Aristotle, so far from denying its permissibility, seems to blame Plato for not using it. Cf. also *Physics* V. 4, 228 b 25, where Aristotle speaks of "the spiral or other magnitude in which any part will not coincide with any other part," and where superposition is obviously contemplated.

Motion without deformation.

It is well known that Helmholtz maintained that geometry requires us to assume the actual existence of rigid bodies and their free mobility in space, whence he inferred that geometry is dependent on mechanics.

Veronese exposed the fallacy in this (*Fondamenti di geometrica*, pp. XXXV–XXXVI, 239–240 note, 615–7), his argument being as follows. Since geometry is concerned with empty space, which is immovable, it would be at least strange if it was necessary to have recourse to the real motion of bodies for a definition, and for the proof of the properties, of immovable space. We must distinguish the intuitive principle of motion in itself from that of motion *without deformation.* Every point of a figure which moves is transferred to another point in space. "Without deformation" means that the mutual relations between the points of the figure do not change, but the relations between them and other figures do change (for if they did not, the figure could not move). Now consider what we mean by saying that, when the figure A has moved from the position A_1 to the position A_2, the relations between the points of A in the position A_2 are unaltered from what they were in the position A_1, are the same in fact as if A had not moved but remained at A_1. We can only say that, judging of the figure (or the body with its physical qualities eliminated) by the impressions it produces in us during its movement, the impressions produced in us in the two different positions (which are in time distinct) *are equal.* In fact, we are making use of the notion of *equality* between two distinct figures. Thus if we say that two bodies are equal when they can be superposed by means of *movement without deformation*, we are committing a *petitio principii.* The notion of the equality of spaces is really prior to that of rigid bodies or of motion without deformation. Helmholtz supported his view by reference to the process of measurement in which the measure must be, at least approximately, a rigid body, but the existence of a rigid body as a standard to measure by, and the question how we discover two equal spaces to be equal, are matters of no concern to the geometer. The method of superposition, depending on motion without deformation, is only of use as a *practical* test; it has nothing to do with the *theory* of geometry.

Compare an acute observation of Schopenhauer (*Die Welt als Wille*, 2 ed. 1844, II. p. 130) which was a criticism in advance of Helmholtz' theory: "I am surprise that, instead of the eleventh axiom [the Parallel-Postulate], the eighth is not rather attacked: 'Figures which coincide (sich decken) are equal to one another.' For *coincidence* (das Sichdecken) is either mere tautology, or something entirely empirical, which belongs, not to pure intuition (Anschauung), but to external sensuous experience. It presupposes in fact the mobility of figures; but that which is movable in space is matter and nothing else. Thus this appeal to coincidence means leaving pure space, the sole element of geometry, in order to pass over to the material and empirical."

Mr Bertrand Russell observes (*Encyclopaedia Britannica*, Suppl. Vol. 4, 1902, Art. "Geometry, non-Euclidean") that the apparent use of motion here is deceptive; what in geometry is called a motion is merely the transference of our attention from one figure to another. Actual superposition, which is nominally employed by Euclid, is not required; all that is required is the transference of our attention from the original figureto a new one defined by the position of some of its elements and by certain properties which it shares with the original figure.

If the method of superposition is given up as a means of defining theoretically the equality of two figures, some other definition of equality is necessary.

But such a definition can be evolved out of *empirical* or *practical* observation of the result of superposing two material representations of figures. This is done by Veronese (*Elementi di geometria*, 1904) and Ingrami (*Elementi di geometria*, 1904). Ingrami says, namely (p. 66):

"If a sheet of paper be folded double, and a triangle be drawn upon it and then cut out, we obtain two triangles *superposed* which we in practice call *equal*. If points A, B, C, D ... be marked on one of the triangles, then, when we place this triangle upon the other (so as to coincide with it), we see that *each* of the particular points taken on the first is superposed on one particular point of the second in such a way that the segments AB, AC, AD, BC, BD, CD, ... are respectively superposed on as many segments in the second triangle and are therefore equal to them respectively. In this way we justify the following

"**Definition of equality.**

"Any two figures whatever will be called *equal* when to the points of one the points of the other can be made to correspond *univocally* [i.e. every *one* point in one to *one distinct* point in the other and *vice versa*] in such a way that the segments which join the points, two and two, in one figure are respectively equal to the segments which join, two and two, the corresponding points in the other."

Ingrami has of course previously postulated as known the signification of the phrase *equal* (*rectilineal*) *segments*, of which we get a *practical* notion when we can place one upon the other or can place a third movable segment successively on both.

New systems of Congruence-Postulates.

In the fourth Article of *Questioni riguardanti le matematiche elementari*, I., pp. 93–122, a review is given of three different systems: (1) that of Pasch in *Vorlesungen über neuere Geometrie*, 1882, p. 101 sqq., (2) that of Veronese according to the *Fondamenti di geometria*, 1891, and the *Elementi* taken together, (3) that of Hilbert (see *Grundlagen der Geometrie*, 1903, pp. 7–15).

These systems differ in the particular conceptions taken by the three authors as primary. (1) Pasch considers as primary the notion of *congruence* or *equality* between *any figures which are made up of a finite number of points only*. The definitions of congruent *segments* and of congruent *angles* have to be *deduced* in the way shown on pp. 102–103 of the Article referred to, after which Eucl. I. 4 follows immediately, and Eucl. I. 26 (I) and I. 8 by a method recalling that in Eucl. I. 7, 8.

(2) Veronese takes as primary the conception of congruence between *segments* (rectilineal). The transition to congruent *angles*, and thence to *triangles* is made by means of the following postulate:

"Let AB, AC and $A'B'$, $A'C'$ be two pairs of straight lines intersecting at A, A', and let there be determined upon them the congruent segments AB, $A'B'$ and the congruent segments AC, $A'C'$;
then, if BC, $B'C'$ are congruent, the two *pairs of straight lines* are congruent."

(3) Hilbert takes as primary the notions of congruence between *both segments and angles*.

It is observed in the Article referred to that, from the theoretical standpoint, Veronese's system is an advance upon that of Pasch, since the idea of congruence between *segments* is more simple than that of congruence between *any figures*; but, didactically, the development of the theory is more complicated when we start from Veronese's system than when we start from that of Pasch.

The system of Hilbert offers advantages over both the others from the point of view of the teaching of geometry, and I shall therefore give a short account of his system only, following the Article above quoted.

Hilbert's system

The following are substantially the Postulates laid down.

(1) *If one segment is congruent with another, the second is also congruent with the first.*

(2) *If an angle is congruent with another angle, the second angle is also congruent with the first.*

(3) *Two segments congruent with a third are congruent with one another.*

(4) *Two angles congruent with a third are congruent with one another.*

(5) *Any segment* AB *is congruent with itself, independently of its sense.*
This we may express symbolically thus:

$$AB \equiv AB \equiv BA.$$

(6) *Any angle* ab *is congruent with itself, independently of its sense.*
This we may express symbolically thus:

$$(ab) \equiv (ab) \equiv (ba).$$

(7) On any straight line r′, starting from any one of its points A′, and on each side of it respectively, there exists one and only one segment congruent with a segment AB belonging to the straight line r.

(8) *Given a ray* a, *issuing from a point* O, *in any plane which contains it and on each of the two sides of it, there exists one and only one ray* b *issuing from* O *such that the angle* (ab) *is congruent with a given angle* (a′b′).

(9) *If* AB, BC *are two consecutive segments of the same straight line* r *(segments, that is, having an extremity and no other point common), and* A′B′, B′C′ *two consecutive segments on another straight line* r′, *and if* AB ≡ A′B′, BC ≡ B′C′, *then*

$$AC \equiv A'C'.$$

(10) *If* (ab), (bc) *are two consecutive angles in the same plane* π *(angles, that is, having the vertex and one side common), and* (a′b′), (b′c′) *two consecutive angles in another plane* π′, *and if*[1] (ab) ≡ (a′b′), (bc) ≡ (b′c′) *then*

$$(ac) \equiv (a'c').$$

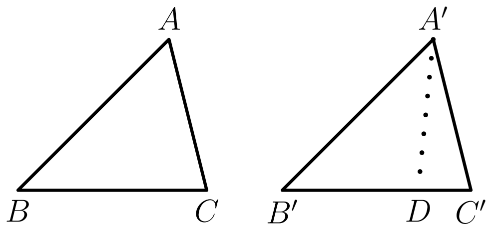
(11) *If two triangles have two sides and the included angles respectively congruent, they have also their third sides congruent as well as the angles opposite to the congruent sides respectively.*

As a matter of fact, Hilbert's postulate corresponding to (11) does not assert the equality of the third sides in each, but only the equality of the two remaining angles in one triangle to the two remaining angles in the other respectively. He proves the equality of the third sides (thereby completing the theorem of Eucl. I. 4) by *reductio ad absurdum* thus. Let ABC, $A'B'C'$ be the two triangles which have the sides AB, AC respectively congruent with the sides $A'B'$, $A'C'$ and the included angle at A congruent with the included angle at A'.

Then, by Hilbert's own postulate, the angles ABC, $A'B'C'$ are congruent, as also the angles ACB, $A'C'B'$.

If BC is not congruent with $B'C'$, let D be taken on $B'C'$ such that BC, $B'D$ are congruent and join $A'D$.

[1][Note added by DRW: the relevant formulae following are printed as $(bc) = (b'c')$ and $(ac) = (a'c')$, i.e., with = in place of the congruence sign ≡ implied by the context.]



Then the two triangles ABC, $A'B'D$ have two sides and the included angles congruent respectively; therefore, by the same postulate, the angles BAC, $B'A'D$ are congruent.

But the angles BAC, $B'A'C'$ are congruent; therefore by (4) above, the angles $B'A'C'$, $B'A'D$ are congruent: which is impossible, since it contradicts (8) above.

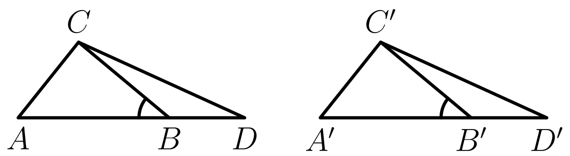
Hence BC, $B'C'$ cannot but be congruent.

Euclid I. 4 is thus proved; but it seems to be as well to include all of that theorem in the postulate, as is done in (11) above, since the two parts of its are equally suggested by empirical observation of the result of one superposition.

A proof similar to that just given immediately establishes Eucl. I. 26 (1), and Hilbert next proves that

If two angles ABC, $A'B'C'$ are congruent with one another, their supplementary angles CBD, $C'B'D'$ are also congruent with one another.

We choose A, D on one of the straight lines forming the first angle, and A', D' on one of those forming the second angle, and again C, C' on the other straight lines forming the angles, so that $A'B'$ is congruent with AB, $C'B'$ with CB, and $D'B'$ with DB.



The triangles ABC, $A'B'C'$ are congruent, by (11) above; and AC is congruent with $A'C'$, and the angle CAB with the angle $C'A'B'$.

Thus AD, $A'D'$ being congruent, by (9), the triangles CAD, $C'A'D'$ are also congruent, by (11);
whence CD is congruent with $C'D'$, and the angle ADC with the angle $A'D'C'$.

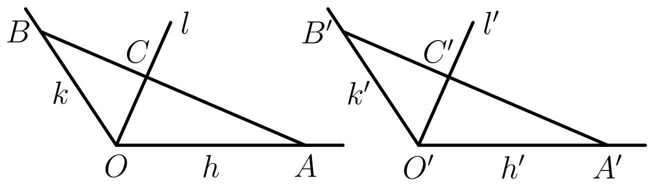
Lastly, by (11), the triangles CDB, $C'D'B'$ are congruent, and the angles CBD, $C'B'D'$ are thus congruent.

Hilbert's next proposition is that

Given that the angle (h, k) *in the plane* α *is congruent with the angle* $(\mathrm{h}', \mathrm{k}')$ *in the plane* α'*, and that* l *is a half-ray in the plane* α *starting from the vertex of the angle* (h, k) *and lying within that angle, there always exists a half-ray* l' *in the second plane* α'*, starting from the vertex of the angle* $(\mathrm{h}', \mathrm{k}')$ *and lying within that angle, such that*

$$(\mathrm{h}, \mathrm{l}) \equiv (\mathrm{h}', \mathrm{l}'), \quad \textit{and} \quad (\mathrm{k}, \mathrm{l}) \equiv (\mathrm{k}', \mathrm{l}').$$

If O, O' are the vertices, we choose points A, B on h, k, and points A', B' on h', k' respectively, such that OA, $O'A'$ are congruent and also OB, $O'B'$.



The triangles OAB, $O'A'B'$ are then congruent; and, if l meets AB in C, we can determine C' on $A'B'$ such that $A'C'$ is congruent with AC.

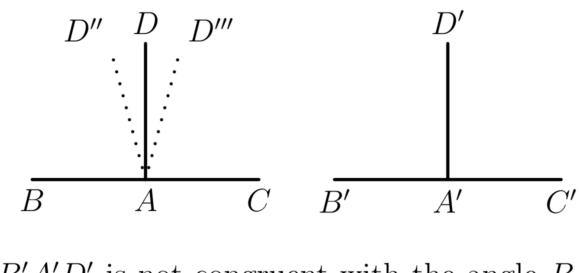
Then l' drawn from O' through C' is the half-ray required.

The congruence of the angles (h, l), (h', l') follows from (11) directly, and that of (k, l) and (k', l') follows in the same way after we have inferred by means of (9) that, AB, AC being respectively congruent with $A'B'$, $A'C'$, the difference BC is congruent with the difference $B'C'$.

It is by means of the two propositions just given that Hilbert proves that

All right angles are congruent with one another.

Let the angle BAD be congruent with its adjacent angle CAD, and likewise the angle $B'A'D'$ congruent with its adjacent angle $C'A'D'$. All four angles are then right angles.



If the angle $B'A'D'$ is not congruent with the angle BAD, let the angle with AB for one side and congruent with the angle $B'A'D'$ be the angle BAD'', so that AD'' falls either within the angle BAD or within the angle DAC. Suppose the former.

By the last proposition but one (about adjacent angles), the angles $B'A'D'$, BAD'' being congruent, the angles $C'A'D'$, CAD'' are congruent.

Hence, by the hypothesis and postulate (4) above, the angles BAD'', CAD'' are also congruent.

And, since the angles BAD, CAD are congruent, we can find within the angle CAD a half-ray CAD''' such that the angles BAD'', CAD''' are congruent, and likewise the angles DAD'', DAD''' (by the last proposition).

But the angles BAD'' and CAD'' were congruent (see above); and it follows, by (4), that the angles CAD'', CAD''' are congruent, which is impossible, since it contradicts postulate (8).

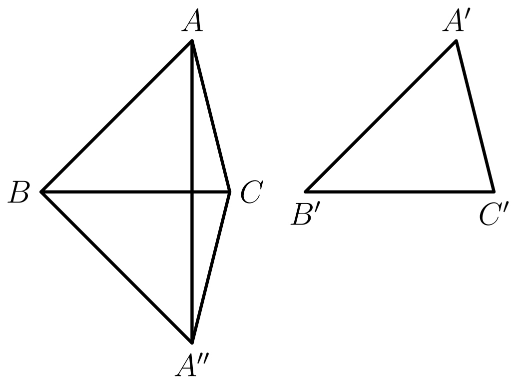
Therefore etc.

Euclid I. 5 follows directly by applying the postulate (11) above to ABC, ACB as distinct triangles.

Postulates (9), (10) above give in substance the proposition that "the sums or differences of segments, or of angles, respectively equal, are equal."

Lastly, Hilbert proves Eucl. I. 8 by means of the theorem of Eucl. I. 5 and the proposition just stated as applied to angles.

ABC, $A'B'C$ being the given triangles with three sides respectively congruent, we suppose an angle CBA'' to be determined, on the side of BC opposite to A, congruent with the angle $A'B'C'$, and we make BA'' equal to $A'B'$.



The proof is obvious, being equivalent to the alternative proof often given in our text-books for Eucl. I. 8.