[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 223–224 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Common Notions 2 and 3.]

Common Notions 2, 3.

2. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.
3. Καὶ ἐὰν ἀπὸ ἴσων ἴσα αφαιρεθῇ, τὰ καταλειπόμενά ἐστὶν ἴσα.

2. *If equals be added to equals, the wholes are equal.*
3. *If equals be subtracted from equals, the remainders are equal.*

These two Common Notions are recognized by Heron and Proclus as genuine. The latter is the axiom which is so favourite an illustration with Aristotle.

Following them in the MSS. and editions there came four others of the same type as 1–3. Three of these are given by Heiberg in brackets; the fourth he omits altogether.

The three are:

(*a*) *If equals be added to unequals, the wholes are unequal.*

(*b*) *Things which are double of the same thing are equal to one another.*

(*c*) *Things which are halves of the same thing are equal to one another.*

The fourth, which was placed between (*a*) and (*b*) was:

(*d*) *If equals be subtracted from unequals, the remainders are unequal.*

Proclus, in observing that axioms ought not to be multiplied, indicates that all should be rejected which follow from the five admitted by him and appearing in the text above (p. 155). He mentions the second of those just quoted (*b*) as one of those to be excluded, since it follows from *Common Notion* 1. Proclus does not mention (*a*), (*c*) or (*d*); an-Nairīzī gives (*a*), (*d*), (*b*) and (*c*), in that order, as Euclid's, adding a note of Simplicius that "three axioms (sententiae acceptae) only are extant in the ancient manuscripts, but the number was increased in the more recent."

(*a*) stands self-condemned because "unequal" tells us nothing. It is easy to see what is wanted if we refer to I. 17, where the same angle is added to a *greater* and a *less*, and it is inferred the the first sum is greater than the second. So far as the wording of (*a*) is concerned, the addition of equal to *greater* or *less* might be supposed to produce *less* and *greater* respectively. If therefore such an axiom were given at all, it should be divided into two. Heiberg conjectures that this axiom may have been taken from the commentary of Pappus, who had the axiom about equals added to unequals quoted below (*e*); if so, it can only be an unskilful adaptation of some remark of Pappus, for his axiom (*e*) has some point, whereas (*a*) is useless.

As regards (*b*), I agree with Tannery in seeing no sufficient reason why, if we reject it (as we certainly must), the words in I. 47 "But things which are double of equals are equal to one another" should be condemned as an interpolation. If they were interpolated, we should have expected to find the same interpolation in I. 42, where the axiom is *tacitly* assumed. I think it quite possible that Euclid may have inserted such words in one case and left them out in another, without necessarily implying either that he was quoting a formal *Common Notion* of his own or that he had *not* included among his Common Notions the particular fact stated as obvious.

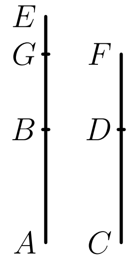
The corresponding axiom (*c*) about the *halves* of equals can hardly be genuine if (*b*) is not, and Proclus does not mention it. Tannery acutely observes however that, when Heiberg, in I. 37, 38, brackets words stating that "the halves of equal things are equal to one another" on the ground that axiom (*c*) was interpolated (although before Theon's time), and explains that Euclid used *Common Notion* 3 in making his inference, he is clearly mistaken. For while axiom (*b*) is an obvious inference from *Common Notion* 2, axiom (*c*) is not an inference from *Common Notion* 3. Tannery says, in a note, that (*c*) would have to be established by *reductio ad absurdum* with the help of axiom (*b*), that is to say, of *Common Notion* 2. But, as the hypothesis in the *reductio ad absurdum* would be that one of the halves is *greater* than the other, and it would therefore be necessary to prove that the one whole is *greater* than the other, while axiom (*b*) or *Common Notion* 2 only refers to *equals*, a little argument would be necessary in addition to the reference to *Common Notion* 2. I think Euclid would not have gone through this process in order to prove (*c*), but would have assumed it as equally obvious with (*b*).

Proclus (pp. 197, 6–198, 5) definitely rejects two other axioms of the above kind given by Pappus, observing that, as they follow from the genuine axioms, they are rightly omitted in most copies, although Pappus said that they were "on record" with the others (συναναγράφεσθαι):

(*e*) *If unequals be added to equals, the difference between the wholes is equal to the difference between the added parts*; and

(*f*) *If equals be added to unequals, the difference between the wholes is equal to the difference between the original unequals.*

Proclus and Simplicius (in an-Nairīzī) give proofs of both. The proof of the former, as given by Simplicius, is as follows:



Let AB, CD be equal magnitudes; and let EB, FD be added to them respectively, EB being greater than FD.

I say that AE exceeds CF be the same difference as that by which BE exceeds DF.

Cut off from BE the magnitude BG equal to DF.

Then since AE exceeds AG by GE, and AG is equal to CF and BG to DF,

AE exceeds CF by the same difference as that by which BE exceeds DF.