[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 190–194 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definition 23.]

Definition 23.

Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

Parallel *straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.*

Παράλληλος (alongside one another) written in one word does not appear in Plato; but with Aristotle it was already a familiar term.

εἰς ἄπειρον cannot be translated "to infinity" because these words might seem to suggest a *region* or *place* infinitely distant, whereas εἰς ἄπειρον, which seems to be used indifferently with ἐπ᾽ ἄπειρον, is adverbial, meaning "without limit," i.e. "indefinitely." Thus the expression is used of a magnitude being "infinitely divisible," or of a series of terms extending without limit.

In both directions, ἐφ᾽ ἠεκάτερα τὰ μέρη, literally "towards both the parts" where "parts" must be used in the sense of "regions" (cf. Thuc. II. 96).

It is clear that with Aristotle the general notion of parallels was that of straight lines *which do not meet*, as in Euclid: thus Aristotle discusses the question whether to think that parallels do meet should be called a geometrical or an ungeometrical error (*Anal. post.* I. 12, 77 b 22), and (more interesting still in relation to Euclid) he observes that there is nothing surprising in different hypotheses leading to the same error, as one might conclude that parallels meet by starting from the assumption, either (a) that the interior (angle) is greater than the exterior, or (b) that the angles of a triangle make up more than two right angles (*Anal. prior.* II. 17, 66 a 11).

Another definition is attributed by Proclus to Posidonius who said that "*parallel lines are those which*, (*being*) *in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other*, while such (straight lines) as make the perpendiculars less and less continually do converge to one another; for the perpendicular is enough to define (ὁρίζειν δύναται) the heights of areas and the distances between lines. For this reason, when the perpendiculars are equal, the distances between the straight lines are equal, but when they become greater and less, the interval is lessened, and the straight lines converge to one another in the direction in which the less perpendiculars are" (Proclus, p. 176, 6–17).

Posidonius' definition, with the explanation as to distances between straight lines, their convergence and divergence, amounts to the definition quoted by

Simplicius (an-Nairīzī, p. 25, ed. Curtze) which described straight lines as parallel *if, when they are produced indefinitely both ways, the distance between them, or the perpendicular drawn from either of them to the other, is always equal and not different.* To the objection that it should be *proved* that the distance between two parallel lines is the perpendicular to them Simplicius replies that the definition will do equally well if all mention of the *perpendicular* be omitted and it be merely stated that the *distance* remains equal, although "for *proving* the matter in question it is necessary to say that one straight line is perpendicular to both" (an-Nairīzī, ed. Besthorn-Heiberg, p. 9). He then quotes the definition of "the philosophier Aganis": "*Parallel straight lines are straight lines, situated in the same plane, the distance between which, if they are produced indefinitely in both directions at the same time, is everywhere the same.*" (This definition forms the basis of the attempt of "Aganis" to prove the Postulate of Parallels.) On the definition Simplicius remarks that the words "situated in the same plane" are perhaps unnecessary, since, if the distance between the lines is everywhere the same, and one does not incline at all towards the other, they must for that reason be in the same plane. He adds that the "distance" referred to in the definition is the shortest line which joins things disjoined. Thus, between point and point, the distance is the straight line joining them; between a point and a straight line, or between a point and a plane it is the perpendicular drawn from the point to the line or plane; "as regards the distance between two lines, that distance is, if the lines are parallel, one and the same, equal to itself at all places on the lines, it is the *shortest* distance and, at all places on the lines, perpendicular to both" (*ibid.* p. 10).

The same idea occurs in a quotation by Proclus (p. 177, 11) from Geminus. As part of a classification of lines which do not meet he observes: "Of lines which do not meet, some are in one plane with one another, others not. Of those which meet and are in one plane, *some are always the same distance from one another*, others lessen the distance continually, as the hyperbola (approaches) the straight line, and the conchoid the straight line (i.e. the asymptote in each case). For these, while the distance is being continually lessened, as continually (in the position of) not meeting, though they converge to one another; they never converge entirely, and this is the most paradoxical thoerem in geometry, since it shows that the convergence of some lines is non-convergent. But of lines which are always an equal distance apart, those which are straight and never make the (distance) between them smaller, and which are in one plane, are parallel."

Thus the *equidistance*-theory of parallels (to which we shall return) is very fully represented in antiquity. I seem also to see traces in Greek writers of a conception equivalent to the vicious *direction*-theory which has been adopted in so many modern text-books. Aristotle has an interesting, though obscure, allusion in *Anal. prior.* II. 16, 65 a 4 to a *petito principii* committed by "those who think they draw parallels" (or "establish a theory of parallels," which is a possible translation of τὰς παραλλήλους γράφειν): "for they unconsciously assume such things as it is not possible to demonstrate if parallels did not exist." It is clear from this that there was a vicious circle in the then current theory of parallels; something which depended for its truth on the properties of parallels was assumed in the actual proof of those properties, e.g. that the three angles of a triangle make up two right angles. This is not the case in Euclid, and the passage makes it clear that it was Euclid himself who got rid of the *petito principii* in earlier text-books by formulating and premising before I. 29 the famous Postulate 5, which must ever be regarded as among the most epoch-making achievements in the domain of geometry. But one of the commentators on Aristotle, Philoponus, has a note on the above passage purporting to give the specific character of the *petito principii* alluded to; and it is here that a *direction*-theory of parallels may be hinted at, whether Philoponus is or is not right in supposing that this was what Aristotle had in mind. Philoponus says: "The same thing is done by those who draw parallels, namely begging the original question; for they will have it that it is possible to draw parallel straight lines from the meridian circle, and they assume a point, so to say, falling on the plane of that circle and thus they draw the straight lines. And what was sought is thereby assumed; for he who does not admit the genesis of the parallels will not admit the point referred to either." What is meant is, I think, somewhat as follows. Given a straight line and a point through which a parallel to it is to be drawn, we are to suppose the given straight line placed in the plane of the meridian. Then we are told to draw through the given point another straight line in the plane of the meridian (strictly speaking it should be drawn in a plane parallel to the plane of the meridian, but the idea is that, compared with the size of the meridian circle, the distance between the point and the straight line is negligible); and this, as I read Philoponus, is supposed to be equivalent to assuming a very distant point in the meridian plane and joining the given point to it. But obviously no ruler would stretch to such a point, and the objector would say that we cannot really direct a straight line to the assumed distant point except by drawing it, without more ado, *parallel* to the given straight line. And herein is the *petito principii.* I am confirmed in seeing in Philoponus an allusion to a *direction*-theory by a remark of Schotten on a similar reference to the meridian plane supposed to be used by advocates of that theory. Schotten is arguing that direction is not in itself a conception such that you can predicate *one* direction of *two* different lines. "If any one should reply that nevertheless many lines can be conceived *which all have the* *direction from north to south*," he replies that this represents only a nominal, not a real, identity of direction.

Coming now to modern times we may classify under three groups practically all the different definitions that have been given of parallels (Schotten, *op. cit.* II. p. 188 sqq.).

(1) *Parallel straight lines have no point common*, under which general conception the following varieties of statement may be included:

(*a*) *they do not cut one another*,

(*b*) *they meet at infinity*, or

(*c*) *they have a common point at infinity.*

(2) *Parallel straight lines have the same, or like, direction or directions*, under which class of definitions must be included all those which introduce transversals and say that the parallels *make equal angles with a transversal.*

(3) *Parallel straight lines have the distance between them constant*; with which group we may connect the attempt to explain a parallel as *the geometrical locus of all points which are equidistant from a straight line.*

But the three points of view have a good deal in common; some of them lead easily to the others. Thus the idea of the lines having no point common led to the notion of their having a common point at infinity, through the influence of modern geometry seeking to embrace different cases under one conception; and then again the idea of the lines having a common point at infinity might suggest their having the same direction. The "non-secant" idea would also naturally lead to that of equidistance (3), since our observation shows that it is things which come nearer to one another that tend to meet, and hence, if lines are not to meet, the obvious thing is to see that they shall not come nearer, i.e. shall remain the same distance apart.

We will now take the three groups in order.

(1) The first observation of Schotten is that the varieties of this group which regard parallels as (*a*) meeting at infinity or (*b*) having a common point at infinity (first mentioned apparently by Kepler, 1604, as a "façon de parler" and then used by Desargues, 1639) as at least unsuitable definitions for elementary text-books. How do we know that the lines cut or meet at infinity? We are not entitled to assume either that they do or that they do not, because "infinity" is outside our field of observation and we cannot verify either. As Gauss says (letter to Schumacher), "Finite man cannot claim to be able to regard the infinite as something to be grasped by means of ordinary methods of observation." Steiner, in speaking of the rays passing through a point and successive points of a straight line, observes that as the point of intersection gets further away the ray moves continually in one and

the same direction ("nach einer und derselben Richtung hin"); only in one position, that in which it is parallel to the straight line, "there is *no real cutting*" between the ray and the straight line; what we have to say is that the ray is "*directed towards the infinitely distant point on the straight line.*" It is true that higher geometry has to assume that the lines do meet at infinity: whether such lines exist in nature or not does not matter (just as we deal with "straight lines" although there is no such thing as a straight line). But if two lines do cut at any finite distance, may not the same thing be true at infinity also? Are lines conceivable which would not cut even at infinity but always remian at the same distance from one another even there? Take the case of a line of railway. Must the two rails meet at infinity so that a train could not stand on them there (whether was could *see* it or not makes no difference)? It seems best therefore to leave to higher geometry the conception of infinitely distant points on a line and of two straight lines meeting at infinity, like *imaginary* points of intersection, and, for the purposes of elementary geometry, to rely on the plain distinction between "parallel" and "cutting" which average human intelligence can readily grasp. This is the method adopted by Euclid in his definition, which of course belongs to the group (1) of definitions regarding parallels as non-secant.

It is significant, I think, that such authorities as Ingrami (*Elementi di geometria*, 1904) and Enriques and Amaldi (*Elementi di geometria*, 1905), after all the discussion of principles that has taken place of late years, give definitions of parallels equivalent to Euclid's: "those straight lines in a plane which have not any point in common are called parallels." Hilbert adopts the same point of view. Veronese, it is true, takes a different line. In his great work *Fondamenti di geometria*, 1891, he had taken a ray to be parallel to another when a point at infinity on the second is situated on the first; but he appears to have come to the conclusion that this definition was unsuitable for his *Elementi.* He avoids however giving the Euclidean definition of parallels as "straight lines in a plane which, though produced indefinitely, never meet," because "no one has ever seen two straight lines of this sort," and because the postulate generally used in connexion with this definition is not evident in the way that, in the field of our experience, it is evident that only one straight line can pass through two points. Hence he gives a different definition, for which he claims the advantage that it is independent of the plane. It is based on a definition of figures "opposite to one another with respect to a point" (or *reflex* figures). "Two figures are opposite ot one another with respect to a point O, e.g. the figures $ABC \ldots$ and $A'B'C' \ldots$, if to every point of the one there corresponds one sole point of the other, and if the segments OA, OB, OC, $\ldots$ joining the points of one figure to O are respectively equal and opposite to the segments OA', OB', OC', $\ldots$ joining to O the corresponding points of the second": then, a *transversal* of two straight lines being any segment having as its extremities one point of one line and one point of the other, "*two straight lines are called parallel if one of them contains two points opposite to two points of the other with respect to the middle point of a common transversal.*" It is true, as Veronese says, that the parallels so defined and the parallels of Euclid are substance the same, but it can hardly be said that the definition gives as good an idea of the essential nature of parallels as does Euclid's. Veronese has to *prove* of course, that his parallels have no point in common, and his "Postulate of Parallels" can hardly be called more evident than Euclid's: "If two straight lines are parallel, they are figures opposite to one another with respect to the middle points of all their transversal segments."

(2) The *direction*-theory.

The fallacy of this theory has nowhere been more completely exposed than by C. L. Dodgson (*Euclid and his modern Rivals*, 1879). According to Killing (*Einführung in die Grundlagen der Geometrie*, I. p. 5) it would appear to have originated with no less a person than Leibniz. In the text-books which employ this method the notion of *direction* appears to be regarded as a primary, not a derivative notion, since no direction is given. But we ought at least to know how the same direction or like directions can be recognised when two different straight lines are in question. But no answer to this question is forthcoming. The fact is that the whole idea as applied to non-coincident straight lines is derived from knowledge of the properties of *parallels*; it is a case of explaining a thing by itself. The idea of parallels being in the *same* direction perhaps arose from the conception of an angle as a *difference* of direction (the hollowness of which has already been exposed); sameness of direction for parallels follows from the same "difference of direction" which both exhibit relatively to a third line. But this is not enough. As Gauss said (*Werke*, IV. p. 365), "If it [identity of direction] is recognized by the equality of the angles formed with *one* third straight line, we do not yet know without an antecedent proof whether this same equality will also be found in the angles formed with a *fourth* straight line" (and any number of other transversals); and in order to make this theory of parallels valid, so far from getting rid of axioms such as Euclid's, you would have to assume as an axiom what is much less axiomatic, namely that "straight lines which make equal corresponding angles with a certain transversal do so with *any* transversal" (Dodgson, p. 101).

(3) In modern times the conception of parallels as *equidistant* straight lines was practically adopted by Clavius (the editor of Euclid, born at Bamberg, 1537) and (according to Saccheri) by Borelli (*Euclides restitutus*, 1658)

although they do not seem to have *defined* parallels in this way. Saccheri points out that, before such definition can be used, it has to be *proved* that "the geometrical locus of points equidistant from a straight line is a straight line." To do him justice, Clavius saw this and tried to prove it: he makes out that the locus is a straight line according to the definition of Euclid, because "it lies evenly with respect to all the points on it"; but there is a confusion here, because such "evenness" as the locus has is with respect to the straight line from which its points are equidistant, and there is nothing to show that it possesses this property with respect to itself. In fact the theorem cannot be proved without a postulate.