[Sir Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd edition), pp. 176–181 (1925).]

[Heath's commentary on Euclid, *Elements*, Book I, Definitions 8, 9.]

Definitions 8, 9.

8. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾿ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

9. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνάιν γραμμαὶ γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

8. *A* plane angle *is the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line.*

9. *And when the lines containing the angle are straight, the angle is called* rectilineal.

The phrase "not in a *straight line*" is strange, seeing that the definition purports to apply to angles formed by *curves* as well as straight lines. We should rather have expected *continuous* (συνεχής) with one another; and Heron takes this to be the meaning, since he at once adds an explanation as to what is meant by lines not being *continuous* (οὐ συνεχεῖς). It looks a though Euclid really intended to define a *rectilineal* angle, but on second thoughts, as a concession to the then common recognition of curvilineal angles, altered "straight lines" into "lines" and separated the definition into two.

I think all our evidence suggests that Euclid's definition of an angle as *inclination* (κλίσις) was a new departure. The word does not occur in Aristotle; and we should gather from him that the idea generally associated with an angle in his time was rather *deflection* or *breaking* of lines (κλάσις): cf. his common use of κεκλάσθαι and other parts of the verb κλᾶν, and also his reference to *one bent line* forming an angle (τὴν κεκαμμενην καὶ ἔχουσαν γωνίαν, *Metaph.* 1016 a 13).

Proclus has a long and elaborate note on this definition, much of which (pp. 121, 12–126, 6) is apparently taken direct from a work by his master Syrianus (ὁ ἡμετερος καθηγεμών). Two criticisms contained in the note need occasion no difficulty. One of these asks how, if an angle be an inclination, one inclination can produce two angles. The other (p. 128, 2) is to the effect that the definition seems to exclude an angle formed by one and the same curve with itself, e.g., the complete *cissoid* [at what we call the "cusp"] or the curve known as the *hippopede* (horse-fetter) [shaped like a lemniscate]. But such an "angle" as this belongs to higher geometry, which Euclid may well be excused for leaving out of account in any case.

Other ancient definitions: Apollonius, Plutarch, Carpus.

Proclus' note records other definitions of great interest. Apollonius defined an angle as *a contracting of a surface or a solid at one point under a broken line or surface* (συναγωγὴ ἐπιφανείας ἢ στερεοῦ πρὸς ἐπὶ σημείῳ ὑπὸ κεκλασμένῃ γραμμῇ ἢ ἐπιφανείᾳ), where again an angle is supposed to be formed by *one* broken line or surface. Still more interesting, perhaps is the definition by "those who say that *the first distance under the point* (τὸ πρῶτον διάστημα ὑηὸ τὸ σημεῖον) *is the angle.* Among these is Plutarch, who insists that Apollonius meant the same thing; for, he says, there must be *some* first distance under the breaking (or deflection) of the including lines or surfaces, though, the distance under the point being continuous, it is impossible to obtain the actual *first*, since every distance is divisible without limit" (ἐπ᾽ ἄπειρον). There is some vagueness in the use of the word "distance" (διάστημα); thus it was objected that "if we anyhow separate off the *first*" (*distance* being apparently the word understood) "and draw a straight line *through it*, we get a *triangle* and not one angle." In spite of the objection, I cannot but see in the idea of Plutarch and the others the germ of a valuable conception in infinitesimals, an attempt (though partial and imperfect) to get at the *rate of divergence* between the lines at their point of meeting as a measure of the angle between them.

A third view of angle was that of Carpus of Antioch, who said "that the angle was a *quantity* (ποσόν), namely a *distance* (διάστημα) between the lines or surfaces containing it. This means that it would be a distance (or divergence) in one *sense* (ἐφ᾽ ἣν διεστώς), although the angle is not on that account a straight line. For it is not everything *extended in one sense* (τὸ ἐφ᾽ ἓν διαστατόν) that is a line." This very phrase "extended one way" being held to define a *line*, it is natural that Carpus' idea should have been described as the greatest possible paradox (πάντων παραδοξότατον). The difficulty seems to have been caused by the want of a different technical term to express a new idea; for Carpus seems undoubtedly to have been anticipating the more modern idea of an angle as representing *divergence* rather than distance, and to have meant by ἐφ᾽ ἓν *in one sense* (*rotationally*) as distinct from *one way* or *in one dimension* (linearly).

To what category does an angle belong?

There was much debate among philosophers as to the particular category (according to the Aristotelian scheme) in which an angle should be placed; is at, namely, a *quantum* (ποσόν), *quale* (ποιόν) or *relation* (πρός τι)?

1. Those who put it in the category of *quantity* argued from the fact that a plane angle is divided by a line and a solid angle by a surface. Since, then, it is a surface which is divided by a line, and a solid which is divided

by a surface, they felt obliged to conclude that an angle *is* a surface or a solid, and therefore a magnitude. But homogeneous finite magnitudes, e.g. plane angles, must bear a ratio to one another, or one must be capable of being multiplied until it exceeds the other. This is, however, not the case with a rectilineal angle and the *horn-like* angle (κερατοειδής), by which latter meant the angle between a circle and the tangent to it, since (Eucl. III. 16) the latter "angle" is less than any rectilineal angle whatever. The objection, it will be observed, assumes that the two sorts of angles *are* homogeneous. Plutarch and Carpus are classed among those who, in one way or another, placed an angle among *magnitudes*; and, as above noted, Plutarch claimed Apollonius as a supporter of his view, although the word *contraction* (of a surface or solid) used by the latter does not in itself suggest magnitude much more than Euclid's *inclination*. It was this last consideration which doubtless led "Aganis," the "friend" (socius) apparently of Simplicius, to substitute for Apollonius' wording "*a quantity which has dimensions and the extremities of which arrive at one point*" (an-Nairīzī, p. 13).

2. Eudemus the Peripatetic, who wrote a whole work on the angle, maintained that it belonged to the category of *quality*. Aristotle had given as his fourth variety of *quality* "figure and the shape subsisting in each thing, and, besides these, straightness, curvature, and the like" (*Categories* 8, 10 a 11). He says that each individual thing is spoken of as *quale* in respect of its form, and he instances a triangle and a square, using them again later on (*ibid.* 11 a 5) to show that it is not all qualities which are susceptible of *more* and *less*; again, in *Physics* I. 5, 188 a 25 *angle*, *straight*, *circular* are called kinds of *figure*. Aristotle would no doubt have regarded *deflection* (κεκλάσθαι) as belonging to the same category with straightness and curvature (καμπυλότης). At all events, Eudemus took up an angle as having its origin in the *breaking* or *deflection* (κλάσις) of lines: deflection, he argued, was quality if straightness was, and that which has its origin in quality is itself quality. Objectors to this view argued thus. If an angle be a quality (ποιότης) like heat or cold, how can it be bisected, say? It can in fact be divided; and if things of which divisibility is an essential attribute are varieties of *quantum* and not qualities, an angle cannot be a quality. Further, the *more* and the *less* are the appropriate attributes of quality, not the equal and the unequal; if therefore angle were a quality, we should have to say of angles, not that one is greater and another smaller, but that one is more an angle and another less an angle, and that two angles are not unequal but *dissimilar* (ἀνόμοιοι). As a matter of fact, we are told by Simplicius, 538, 21, on Arist. *De caelo* that those who brought the angle under the category of *quale* did call equal angles *similar* angles; and Aristotle himself speaks of *similar* angles in this sense in *De caelo* 296 b 20, 311 b 34.

3. Euclid and all who called an angle an inclination are held by Syrianus to have classed it as a *relation* (πρός τι). Yet Euclid certainly regarded angles as magnitudes; this is clear both from the earliest propositions dealing specifically with angles, e.g. 1. 9, 13, and also (though in another way) from his describing an angle in the very next definition and always as *contained* (περιεχομένη) by the two lines forming it (Simon, *Euclid*, p. 28).

Proclus (i.e. in this case Syrianus) adds that the truth lies between these three views. the angle partakes in fact of all those categories: it needs the *quantity* involved in magnitude, thereby becoming susceptible of equality, inequality and the like; it needs the *quality* given it by its *form*, and lastly the *relation* subsisting between the lines or planes bounding it.

Ancient classification of "angles."

An elaborate classification of angles given by Proclus (pp. 126, 7–127, 16) may safely be attributed to Geminus. In order to show it by a diagram it will be necessary to make a convention about terms. Angles are to be understood under each class, "line-circumference" means an angle contained by a straight line and an arc of a circle, "line-convex" an angle contained by a straight line and a circular arc with convexity *outwards*, and so in in every case.

- Angles
  - on surfaces
    - on *simple* surfaces
      - on *planes*
        - made by *simple* lines
          - line-line
          - line-circumf.
            - line-convex (e.g. angle of a semicircle)
            - line-concave (e.g., *horn-like* (κερατοειδής)
          - circumf.-circumf.
            - convex-convex (ἀμφίκυρτοι)
            - concave-concave (ἀμφίκοιλοι) or "scraper-like" (ξυστροειδεῖς)
            - mixed, or convex-concave (e.g. those of *lunes*)
        - made by "*mixed*" lines e.g. the angle made by a curve, such as the *cissoid* and *hippopede*, with itself)
        - by one of each (e.g. the angle formed by an ellipse and its axis or by an ellipse and a circle)
      - on spherical surfaces
    - on *mixed* surfaces (e.g. cones, cylinders)
  - in solids (ἐν στερεοῖς)

Definitions of angle classified.

As for the point, straight line, and plane, so for the *angle*, Schotten gives a valuable summary, classification and criticism of the different modern views up to date (*Inhalt und Methode des planimetrischen Unterrichts*, II., 1893, pp. 94–183); and for later developments represented by Veronese reference may be made to the third article (by Amaldi) in *Questioni riguardanti le matematiche elementari*, I. (Bologna, 1912).

With one or two exceptions, says Schotten, the definitions of angle may be classed in three groups representing generally the following views:

1. *The angle is the difference of direction between two straight lines.* (With this group may compared Euclid's definition of an angle as an inclination.)

2. *The angle is the quantity or amount* (*or the measure*) *of the rotation necessary to bring one of its sides from its own position to that of the other side without moving out of the plane containing both.*

3. *The angle is the portion of a plane included between two straight lines in the plane which meet in a point* (*or two rays issuing from the point*).

It is remarkable however that nearly all the text-books which give definitions different from those in group 2 add to them something pointing to a connexion between an angle and rotation: a striking indication that the essential nature of an angle is closely connected with rotation, and that a good definition must take account of that connexion.

The definitions in the first group must be admitted to be tautologous, or *circular*, inasmuch as they really presuppose some conception of an angle. *Direction* (*as between two given points*) may no doubt be regarded as a primary notion; and it may be defined as "the immediate relation of two points which the *ray* enables us to realise" (Schotten). But "a *direction* is no intensive magnitude, and therefore two directions cannot have any quantitative difference" (Bürklen). Nor is direction susceptible of differences such as those between qualities, e.g., colours. Direction is a *singular* entity: there cannot be different *sorts* or *degrees* of direction. If we speak of "a *different* direction," we use the word equivocally; what we mean is simply "another" direction. The fact is that these definitions of an angle as a difference of direction unconsciously appeal to something outside the notion of direction altogether, to some conception equivalent to that of the angle itself.

Recent Italian views.

The second group of definitions are (says Amaldi) based on the idea of the rotation of a straight line or ray in a plane about a point: an idea which, logically formulated, may lead to a convenient method of introducing the angle. But it must be made independent of *metric* conceptions, or of the conception of *congruence*, so as to bring out *first* the notion of an angle, and *afterwards* the notion of *equal* angles.

The third group of definitions satisfy the condition of not including metric conceptions; but they do not entirely correspond to our intuitive conception of an angle, to which we attribute the character of an entity in *one* dimension (as Veronese says) with respect to the *ray* as element, or an entity in *two* dimensions with reference to *points* as elements, which may be called an *angular sector*. The defect is however easily remedied by considering the angle as "the aggregate of the rays issuing from the vertex and comprised in the angular sector."

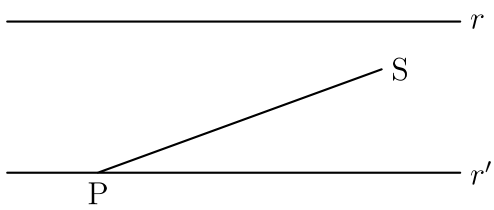
Proceeding to consider the principal methods of arriving at the logical formulation of the first superficial properties of the *plane* from which a definition of the angle may emerge, Amaldi distinguishes two points of view (1) the *genetic*, (2) the *actual*.

(1) From the first point of view we consider the *cluster of straight lines* or *rays* (the aggregate of all the straight lines in a plane passing through a point, or of all the rays with their extremities in that point) as generated by the movement of a straight line or ray in the plane, about a point. This leads to the *postulation* of a *closed order*, or *circular disposition*, of the straight lines or rays in a cluster. Next comes the connexion subsisting between the disposition of any two clusters whatever in one plane, and so on.

(2) Starting from the point of view of the *actual*, we lay the foundation of the definition of an angle in *the division of the plane into two parts* (half-planes) *by the straight line.* Next two straight lines (a, b) in the plane, intersecting at a point O, divide the plane into four *regions* which are called *angular sectors* (convex); and finally the *angle* (ab) or (ba) may be defined as *the aggregate of the rays issuing from* O *and belonging to the angular sector which has* a *and* b *for sides.*

Veronese's procedure (in his *Elementi*) is as follows. He begins with the first properties of the plane introduced by the following definition.

The figure given by all the straight lines joining the points of a straight line r to a point P outside it and by the parallel to r through P is called a *cluster of straight lines*, a *cluster of rays*, or a *plane*, according as we consider the *element* of the figure itself to be the *straight line*, the *ray* terminated at P, or a *point.*

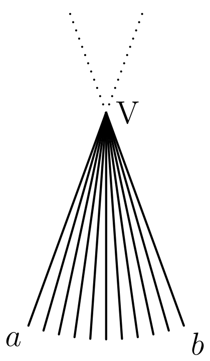


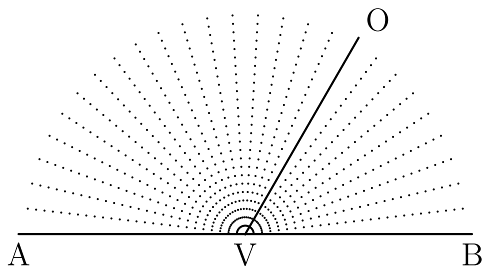
[It will be observed that this method of producing a plane involves using the *parallel* to r. This presents no difficulty to Veronese because he has previously define parallels, without reference to the plane, by means of *reflex* or *opposite* figures, with respect to a point O: "two straight lines are called *parallel*, if one of them contains two points opposite to (or the reflex of) two points of the other with respect to the middle point of a common traversal (of the two lines)." He proves by means of a postulate that the parallel r' does belong to the plane Pr. Ingrami avoids the use of the parallel by defining a *plane* as "the figure formed by the half straight lines which project from an internal point of a triangle (i.e. a point on a line joining any vertex of a *three-side* to a point of the opposite side) the points of its perimeter," and then defining a *cluster* of rays as "the aggregate of the half straight lines in a plane starting from a given point of the plane and passing through the points of the perimeter of a triangle containing the point."]

Veronese goes on to the definition of an angle. "*We call an* angle *a part of a cluster of rays, bounded by two rays* (*as the segment is a part of a straight line bounded by two points*).

"*An angle of the cluster, the bounding rays of which are opposite, is called a* flat angle."

Then, after a postulate corresponding to postulates which he lays down for a *rectilineal segment* and for a *straight line*, Veronese proves that *all flat angles* are equal to one another.





Hence he concludes that "the cluster of rays is a homogeneous linear system in which the element is the *ray* instead of the *point*. The cluster being a homogeneous linear system, all the propositions deduced from [Veronese's] Post. 1 for the straight line apply to it, e.g. that relative to the sum and difference of the segments: it is only necessary to substitute the ray for the point, and the angle for the segment."